Mathematical Reasoning, Edited by Lyn D. English

Ed Dubinsky, Georgia State University

This book explores how people think mathematially and how they learn to do so. Taken together, the 13 separate chapters by different authors consider the full range of mathematical thinking from early childhood through the school years, university levels, and up to the threshold of research. To what extent can mathematical thought be analyzed in terms of analogy, metaphor, and imagery, and how can such an analysis help us improve mathematics education?

Regarding the latter, the reader who is looking for prescriptions for what to do in class next week will be disappointed. Several of the chapters do present examples of effective teaching that implement the ideas they are presenting, and there are indications of potential pitfalls. The basic message of this and any investigation into mathematical thinking however, is as Presmeg (p. 277) points out, that teachers can gain "...a heightened understanding of some of the processes involved in learning and in doing mathematics" and "...might enhance their apprehension of the difficulties which their students might be experiencing in these processes." My own view is that, in the long run, deepening our understanding of the nature of mathematical thought is going to be more beneficial to our effectiveness as teachers than reading about classroom examples we might appropriate.

In reading this book, then, for what it might tell us about mathematical reasoning, I would advise the reader to go quickly past the introduction. Its main point is to try to explain what might be meant by analogy, metaphor and imagery, but it does not succeed very well. There is a definition of analogy (p. 5) and also of metaphor (p. 7). These two mechanisms for thinking are supposed to be different, but I can find absolutely no difference between the two statements. Sfard does a much better job of distinguishing them in a later chapter and I will have more to say about that below.

The 12 chapters that make up the body of this book are organized like a sandwich. The opening and closing pieces are theoretical reflections on how metaphor can be used as a device to describe the contents of mathematics (Lakoff & Nuñez, Chapter 2) and to describe where this content comes from (Sfard, Chapter 13). The remaining 10 chapters are individual essays on various aspects of these two general themes.

As with many sandwiches, the bread is more interesting than the innards. Although I disagree with much of what Lakoff and Nuñez and Sfard have to say, both essays address extremely important questions and the authors have made serious attempts to develop and support coherent positions which they present in a very readable manner. I consider these two chapters to make serious contributions to the philosophy of mathematics and I recommend that they be read and thought about seriously. In this review, I will concentrate on these two chapters.

The chapters in between were, for me, somewhat less interesting. For completeness' sake, I will briefly summarize their contents. Davis and Maher remind us that "new ideas come from old ideas" but they don't tell us much about the mechanisms by which such constructions are made nor do they tell us how to avoid the "Learning Paradox" by which nothing new can be learned since it must already be present in the foundation. (see Piattelli-Palmarini, 1980, for a discussion by Fodor who uses this point to argue for an innatist theory of knowledge and also Bereiter, 1985). There are four chapters concerned with how children do or do not use analogy to solve mathematical problems. Some of the authors are convinced that very young children show evidence of analogical reasoning, others suggest that this does not really happen very much, or very effectively at any age. The data from research is certainly very mixed. These chapters are followed with a summary by Ratterman who tries to give a theoretical foundation to their content. In her first chapter, Presmeg offers explanations of the meanings of several terms used often in the book: metaphors, metonymies and semiotics. Wheatley argues that imagery is important for successful problem solving and in a second chapter, Presmeg considers the interplay between visual and analytical thinking. Finally, Clements and Sarama consider the value of computer programming, or specifically, programming in Logo, in helping children develop an understanding of mathematical concepts. This chapter seems at best loosely related to the rest of the book.

Let me turn now to the two pieces of bread in this sandwich, the opening chapter by Lakoff and Nuñez and the final, summarizing chapter by Sfard. In my opinion, these are the two most important chapters of the book.

Metaphor as a structuring mechanism for mathematics.

The basis for Lakoff and Nuñez' essay is their oft repeated criticism that much of 20^{th} century mathematics is not about ideas, but rather is mainly concerned with symbols, their arrangements according to the rules of syntax, and model-theoretical interpretations of these formal statements. This, they seem to suggest, is why modern mathematical treatments of topics such as limits, continuity, infinity, etc. are so inaccessible to most students. Contemporary mathematics, they argue, has gotten away from concrete phenomena, and many standard ways of thinking about mathematics should be replaced by a system of metaphors.

No one can doubt the existence of serious dissonance between much of mathematics and direct, everyday experience, as well as the importance of finding ways to deal with this gap if mathematics is to survive as a full fledged part of our culture and not merely glass bead mind games played by a select few. The authors are to be congratulated for a serious attempt to develop a philosophical foundation for closing the gap. The solution of Lakoff and Nuñez is to "bring embodied human minds, as they have come to be understood recently in cognitive science, back into mathematics." They do this by means of metaphors, and their idea is that every mathematical concept can be seen to come directly from some physical, everyday experience ("grounding metaphors") or can be linked to such, directly or through a sequence of connections ("linking metaphors".)

For example, arithmetic concepts are grounded by metaphors such as "Numbers are collections of physical objects of uniform size", "Numbers are physical objects", and "Numbers are locations on a path." The concept of function is grounded by "A function is a machine" and "A function is a collection of objects with directional links." The function concept is also linked to the concept of variable (grounded by the metaphor of a traveler) by the metaphor that a function is a traveler whose motion is determined by two variable-travelers. This same traveler metaphor then is a linking metaphor for the limit concept.

Lakoff and Nuñez's effort is to be applauded for several reasons. They are making a systematic attempt on both general and specific levels to work out a consistent epistemology of mathematics. Unlike many discussions of the philosophy of mathematics, this one concentrates the examples on mathematics and does not restrict consideration to the most elementary concepts. Of course numbers and arithmetic are major topics, but the authors also discuss more advanced concepts such as limits, continuity, infinity, and space-filling curves.

I am unconvinced, however, that this admirable attempt works. I think that there are a number of difficulties with their analysis, that there is very little indication of how this particular epistemology might help in teaching the mathematical concepts they consider, and most seriously, that they are misunderstanding a number of the ways of thinking about mathematics that they are rejecting.

Difficulties with their analysis

I find some of the metaphors questionable. For example, the arithmetic of functions is an important mathematical idea that can be very difficult for undergraduates to understand. The Lakoff and Nuñez metaphor places the concept of the sum of two given functions, f and g, entirely in the arithmetic operations on the numbers in the range of these two functions, and their epistemology does not deal at all with the existence of $f + g$ as an object. They offer two metaphors for continuity — gaplessness and preservation of closeness, but I am unable to see how one might build on these to move, for example, to the idea of uniform continuity. The fundamental concept of zero is grounded in the metaphor of an empty collection. Logically this makes sense, but several decades of teaching makes me wonder for whom this metaphor captures the notion of zero.

These and many other examples bring into question the sources of the metaphors that are offered. Do they really form a natural basis for our thinking, or are they the logical creations of the authors who are trying to develop a consistent epistemology? The only "evidence" they present for the former is a number of standard phrases in ordinary discourse. This assumes that a given linguistic expression can be taken at face value and does not depend on the interpretations of the person who made the statement, of the epistemologist who selected it as an example, and of the reader of the analysis. All of whose interpretations may be different.

I find some gaps in their reasoning. For example, they misinterpret standard conventions of mathematics connected with expressions such as $\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$ and $\lim_{x\to 0}\left(\frac{1}{x}\right) = \infty$, missing the point that the first asserts an equality of two numbers, but the second is just a notational convention for a certain phenomenon and does not mean that ∞ is supposed to be a number. There is a long discussion of the continuity and differentiability at 0 of functions defined by expressions such as $x^n \sin\left(\frac{1}{x}\right)$. They seem to deal with these pictorially in terms of gaps and tangents and directions. But although their descriptions work for $n = 1, 2$ they break down for larger values of n. In my view these are not "monster" functions as Lakoff and Nuñez designate them, but rather an intriguing collection of examples that serve to ground the concepts of continuity and differentiability of different orders in ways that do not seem to be included in their metaphor epistemology.

Finally, I think that the authors miss much of the point of Cantor's theory of infinite sets. It may be that our notion of "same number" is grounded for finite sets in ways such as the authors suggest. But the point of Cantor's analyses is that these intuitions don't work for infinite sets and new intuitions must be constructed. It is the difficulty in revising intuitions and our lack of understanding of how teachers can help that has produced the "confusion for generations of students" to which Lakoff and Nuñez refer. They suggest that the cause is that the point has never been explicitly stated. It is increasingly our experiences as teachers that serious conceptual difficulties of students are barely affected by explicit statements of any kind. Much more is needed than stating things explicitly. This brings us to a consideration of the metaphor epistemology and teaching.

Relations to teaching mathematics

Lakoff and Nuñez list three goals for their entire enterprise, and the third of these is "As a helpful task for mathematics education." I see very little in what they tell us that would be helpful. Their metaphor for limit of a function, for example, is about two travelers approaching certain locations. What is the teacher to do with this idea? Of course, countless efforts are made by all teachers to link the concept with this and other metaphors. One of the more interesting is the ϵ-δ contract of Courant and Robbins, 1996. The fact of the matter is that such discussions seem to provide very little help and apparently make almost no difference in student learning. Indeed, I don't see any way to use a metaphor epistemology other than to explain the metaphors to the students. I don't think this is adequate, and I am disappointed that Lakoff and Nuñez pay so little attention to this particular one of their goals.

I would propose as an alternative a constructivist epistemology. By constructivist, I do not mean that students are expected to discover mathematical concepts on their own without help from the teacher, although the opportunity for students to *try* to discover mathematical ideas can be an important pedagogical tool, if used appropriately. By constructivist I mean the idea that learning mathematics requires an individual to construct mathematical understandings in her or his own mind. This can happen as a result of experiences that are provided in a learning situation and that include, but are not restricted to, hearing explanations. Of course these understandings the individual constructs must be consistent with the understandings held by mathematicians, but they cannot be *given* by one person to another. They must, ultimately, come from the learner, and the teacher can only provide opportunities, stimuli and support.

The point of a constructivist, as opposed to a metaphor, epistemology is that there is at least the hope that if learning mathematics occurs through making mental constructions, we can try to find out what these constructions might be and look for specific ways (such as writing computer programs, working on tasks in groups, writing essays, etc.) to help students make specific constructions. There is, in fact, a considerable amount of work being done along these lines, but that is a matter for another story.

Metaphor as a tool for constructing mathematics

The metaphor is, for literature, an extremely important, indeed essential, mechanism and it is certainly true, as Sfard contends in her essay, that "metaphor is pervasive in everyday life." Therefore it is not only tempting, but necessary, to consider the question of what role the metaphor plays in the development of mathematical understanding and how important it is. In fact, it is Sfard's contention that "Mathematics is not less dependent on metaphor than literature is." In asserting the "ubiquity of metaphor and its power to create for us the world we live in—including the most remote and esoteric regions of abstract mathematics" she echoes the thesis of Lakoff and Nuñez.

Sfard not only develops this thesis but relates it to existing theories of the development of mathematical knowledge. In particular, she considers theoretical analyses of mathematical understanding, developed by herself and others, in terms of what is called an operational-structural or process-object dialectic. An open question in Sfard's version of this theory has been to understand the mechanism of, for example, passing from operational to structural, that is, from process to object understandings of a mathematical concept. Sfard offers the mechanism of metaphor as an answer to this question and as a result, her chapter tends to unify the somewhat disparate discussions in the other 12 chapters of the book. In particular she clarifies the distinction between analogy and metaphor in that the former is a linking of two concepts already constructed and the latter is the use of an existing concept to construct a new one. Another contribution of this chapter is to situate the entire book within an epistemological development that has been taking place for more than a decade.

The next question then, is how exactly does one use metaphor to create meaning? Sfard's answer is, language.

Language as the mechanism

In Sfard's view, "…the attention to metaphor is an attention to language." She argues that it would be impossible to have a metaphor-free language and suggests that the use of language can lead to the construction of mathematical meaning. Accepting the idea of Davis and Maher that new mathematical ideas are constructed from old ones (both in mathematical research and in learning mathematics), Sfard asserts that new symbols usher in new mathematical concepts, that the meaning of a new symbol is inherited from "the meaning of the symbols whose linguistic footsteps it is following."

Consider the particular case of understanding a mathematical process as an object, perhaps the most difficult of all mental constructions. According to Sfard, "what we call 'mathematical objects' are metaphors resulting from certain linguistic transplants." She asserts that "…introduction of a new mathematical symbol is often enough to … bring about reification" (the encapsulation of a process into an object.) Indeed, she believes, "If we write a new name or a new symbol in the slot reserved for objects—the new signifier will eventually bring about an emergence of a new mathematical object."

The case for the role of language and metaphor

In this reviewer's opinion, the arguments presented are too weak to justify the contention that it is through the use of language that we construct new mathematics. Sfard points out that you can't explain the phenomenon of abstraction without a reference to metaphorical projection. This may be true, and although she gives many examples of mathematical discourse that is full of metaphor, her evidence does not go beyond these illustrations. But the fact that we may use metaphors to *explain* abstraction does not imply that we use metaphors to *perform* abstraction.

Furthermore, it not clear how the use of metaphor in mathematical discourse can help students learn mathematics. On the one hand, Sfard seems to be suggesting that students learn from the teacher who, like the parent teaching the child the proper words for various objects, introduces specific linguistic constructs (e.g., nouns to replace verb phrases) and this is supposed to lead the student to construct objects. On the other hand, however, she reminds us of the ineffectiveness of introducing terminology for a concept before the learners have constructed the concept. So how does Sfard think this construction is to be made? Is there a contradiction, or at least a gap here? It seems to me that something more than a linguistic mechanism is needed to help students construct new mathematical ideas.

Suppose we were to accept Sfard's contention that mathematical meaning is constructed by engaging in discourse that uses certain kinds of words in certain ways. This seems to suggest that teaching mathematics should consist in talking about ideas using appropriate linguistic structures. Many mathematicians today, however, are concluding from their experiences that the pedagogical approach of explaining mathematics to students, or lecturing, is far from adequate. Sfard herself has followed this line and concluded that sophisticated mathematical concepts (such as those involving objects that must be constructed from processes) may not be accessible to very many students (Sfard, 1992, 1994.) I find this very disappointing because it seems to ignore the progress that has been made in recent years in going beyond what Sfard acknowledges are the limitations of the metaphorical mechanism.

Beyond linguistic mechanisms

Consider the example of negative numbers. How does $3-8$ become an object? For Sfard it begins as "impossible subtraction" and becomes a new object by metaphorical projection of the existing concept of number because such expressions "may be manipulated in ways similar to those in which the 'regular' numerical expressions are operated on." An alternative is to consider that $3-8$ is not an impossible subtraction but a reversal of the process of adding 8. Here *reversal* is a general mechanism used to create mental processes. Another general tool I have already mentioned is encapsulation by which a process becomes an object in an individual's mind. It turns out that this kind of interpretation can be made for a wide variety of mathematical concepts such as the notion of function as object, the analysis of cosets to arrive at Lagrange's Theorem, and the construction of quotient groups.

Can such an epistemological approach lead to pedagogical strategies that are more effective than using metaphors in mathematical discourse? The answer seems to be yes and this relates to the alternative discussed above to the epistemology of Lakoff and Nuñez. In both cases my suggestion is that formalism can be more effective than metaphor as a mechanism for constructing meaning. There are methodologies by which students' negative reactions to formal expressions can be overcome. These methods are not unrelated to those suggested by Clements and Serama in their chapter of this book. It turns out that performing certain computer tasks and reflecting on the operation of the computer in these tasks can help students use even complex formal statements (e.g., involving two and three level quantifications) to construct mental processes and encapsulate them into objects. The results, in terms of learning mathematics, are very encouraging. (See, for example, Dubinsky, 1995, 1997a,b.)

The alternative I am suggesting may be more difficult and less dramatic than the striking analogies and mechanisms of literature, but I think that in the long run, the harder approach may be the more effective.

1 References

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