After Examples and before Proofs: Constructing Mental Objects

There is a general feeling that Mathematics consists mainly of specific examples (embodied in pictures, algorithms for calculation, etc.) and proofs. There is a highly justified discomfort with a mathematics curriculum that focuses on the former, and serious disagreement about when and where the latter should come to the fore. I happen to be one of those who believes that serious concern for proofs should come later (in the last year or two of the undergraduate curriculum) rather than sooner. Now it may seem that this is a call for watering down, but I see it as an attempt to make the real meeting of the student with the subject happen at a higher level of sophistication.

Consider what happens when a student who is quite successful with the example and calculation aspect of mathematics first comes into contact with proofs, somewhere between the last year of high school and the first year or two of college. With the exception of the very few students with special talent for mathematics (of whom we are not speaking here) essentially nothing intellectual happens and the student reverts, as do the majority according to Ed Moise [2], to experiencing mathematics as a "reportoire of imitative behavior patterns."

I think there is a tertium between calculations and proofs that would fit in very nicely between the two. It would result in an increase in the level of mathematics that the student really does relate to, and at the same time prepare the way for a later concentration on proofs.

What is this tertium? I suggest that the (mental) construction of mathematical objects can play this role. Most undergraduates see a function as a process of doing something as opposed to an object to which something is done. This is why the notion that "the derivative of a function is a function" seems so foreign. Deep mathematical understanding of an idea such as the *statement* of the fundamental theorem of calculus require that a function be interpreted both as a process and as an object.

Another example comes from abstract algebra and the notion of coset. For many students in a first semester course in abstract algebra, a coset is a process of forming a set in a certain way. It is not an object and therefore, counting the number of cosets, comparing their cardinalities, defining a binary object on them makes absolutely no sense. This is why the proof of Lagrange's theorem and the construction of quotient groups cause so much trouble.

Examples abound in mathematics of ideas which can only be understood if you are able to interpret a process as an object and vice-versa. Most of these examples, like the two I just gave, are closely connected to proofs with which students have serious difficulties. There are now pedagogical strategies, using computer programming that can help students construct these objects in their minds and there are indications that using these strategies might help students display a better understanding of proofs of theorems like Lagrange's theorem [1]. This amounts to a very different use of technology with a very different set of goals than geometry software or computer algebra systems, but it can be at least as effective.

Our curriculum generally asks students to understand logical arguments establishing relations between mathematical objects that are not, for the students, objects at all. I think that if we would pay attention to the mental construction of objects it would provide a rich set of experiences in the last years of high school and first years of college. The result, I believe, would be that by the end of the four years of college, a noticeably larger group of students would have a significantly deeper understanding of mathematics, including proofs, than we see today.

References

1. Leron, U. and E. Dubinsky, *An Abstract algebra Story*, American Mathematical monthly, **102**, 3, (March, 1995) 247-272.

2. Moise, E.E. *Mathematics, computation, and psychic intelligence* in "V.P. Hansen & M.J. Zweng (Eds.), **Computers in mathematics education**," (1984 Yearbook of the National Council of Teachers of Mathematics, pp. 35-42. Reston, VA : NCTM (1984).

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