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# Computers in Teaching and Learning Discrete Mathematics and Abstract Algebra

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Perhaps the most important point to make today about the use of computers in helping students learn mathematics is that computers are not enough. Not even the addition of special software will suffice to make the significant improvement in mathematical understanding on the part of our population that the rapidly emerging technological society of the next few decades will require.

Indeed, the hardware and software systems now available boggle the mind of someone like me who, hardly more than three decades ago, first encountered the world of computers and found that you had to code everything (by hand) into zeros and ones, and punch holes in paper tape in order to communicate with a conglomeration of flip-flops, wires and vacuum tubes that would not fit in this building.

Not only are we, each decade, seeing computer power exponentially greater than the last, but much of this is available for use in mathematics courses. State of the art hardware, general purpose educational sofware systems, and special packages for learning one or another mathematics topic abound in the marketplace and also the readily available "public domain".

It is, of course, much too early to pass any judgements on how well we are doing with all of these wonderful tools. I think that people are beginning to consider the possibility that the difference they are making is far less than we had hoped. I should like to offer the prediction that this will be increasingly the case and will continue to be so as long as we rely on hardware and software alone.

What I think needs be added to the equation is an understanding of how people learn the mathematics we would like them to understand. In my opinion the things that we normally do as teachers are not the kinds of things that are likely to help people learn. They do not reflect the actual learning process as it relates to mathematics. I think that whether we make use of technology or not, the overall situation in mathematics education will not change very much unless our instructional treatments become meaningful in terms, not of how we are used to teaching, but in terms of how people actually learn mathematics.

This paper describes a portion of a project which attempts to study the learning process in mathematics and to develop instructional treatments that "speak to" that process. Our project includes the ongoing development of a theory of learning mathematics, empirical research, and the use of computers. There is only space here to describe a small part of the activities we have been engaged in, but the interested reader can look at the papers that we refer to.

I will begin the discussion of our work with a description of some specific conceptual difficulties in certain mathematical topics that we and others have observed in students. Next I give an indication of the learning theory that is driving our development. Then I propose some explanations of the difficulties based on this theory. Finally, I describe some instructional treatments that we have designed, using computers, that are based on our explanations. I am only able to refer, in the conclusion, to papers where we have described the results of using these treatments in classroom situations. Suffice it to say here that our results have convinced us that this is a very promising approach to helping students learn mathematics.

# 1 Some Conceptual Difficulties for Students

## 1.1 Functions

It can be argued that functions form the single most important idea in all of mathematics, at least in terms of understanding the subject as well as for using it. Knowing what a function is, how to determine its properties, and being able to recognize the (potential) role of the function concept in organizing one's thinking about a problem situation, are all esssential aspects of mathematical thought.

A great deal of research has been done laying out a number of difficulties students have with the function concept [1,2,3,5,16,17,19]. I will add to that a few, possibly new, examples from our own work. I categorize these examples in terms of processes and objects because, as we shall see, this is how they are treated by our theory. The category of graphs is included because of the widespread feeling that their role is essential in understanding the concept of function.

### 1.1.1 Processes

We have found that many students require the presence of an explicit formula before they are willing to agree that a function is present. For example, in giving students a two-column table of "dues owed" in a club, many subjects that were interviewed tried to make up a formula that computed the amount owed and, when they did not succeed, insisted that the situation could not be described in terms of a function. (See [2] for details.)

Another example has to do with implicitly defined functions. For example, given the following two equations

$$y^4 = x^3$$

$$2x^3y - \sqrt{x}\log y = 2$$

students were much more likely to conclude that the first defined a function than that the second did. In the former case, they either restricted $y$ to positive values and took the fourth root to solve for $y$ in terms of $x$, or they let $y$ stand for the independent variable and solved for $x$. For the second equation, many said they had not succeeded in solving for one variable in terms of the other and so could not tell if there was a function. When they were asked what would be their conclusion if they became convinced that they could not ever succeed in finding an expression, they said if that were certain, then there is no function [2].

One other kind of example runs through the literature, going at least as far back as Euler who said that an expression such as

$$V(t) = \begin{cases} 26.7t^2 & \text{if } 0 < t \leq 50 \\ (4/3)\pi t^3 & \text{if } 50 < t \end{cases}$$

was not a function, but rather represented two functions [15]. Many students today agree very strongly with Euler. Perhaps more serious is that whether or not students can understand the sense in which such an expression can specify a function, they are unable to perform essential operations with functions of this kind such as composing two of them, or computing the derivative at the seam.

Given such difficulties, we should not be surprised that students are unlikely to succeed with a problem of the following kind.

> Given the fact that three functions $F$, $G$, and $H$ satisfy the relation
>
> $$H = F \circ G$$
>
> and given particular functions, $H$ and $F$, find $G$.
>
> Or, alternatively, the same problem in which $H$ and $F$ are given and the task is to find $G$.

Explicit examples of such problems and students' difficulties with them are given in [1].

#### 1.1.2 Objects

The *algebra of functions* is a technical term for the idea of treating functions as objects and performing operations on them such as adding, subtracting, multiplying, and composing. It appears that students have great difficulty in moving from the domain of manipulating numbers to the "higher plane", to use a Piagetian term [18], of performing essentially the same manipulations on functions.

Indeed, not only students but many high school and even college mathematics teachers have difficulty with the idea of an operation (for example a computer program) producing for an answer, not a number, but a function. I have witnessed college teachers having considerable difficulty understanding the following computer procedure and evaluation statement. It is understood that `f` refers to a previously defined procedure implementing a mathematical function.

```
k := func(p);
        if is_func(p) then
            return func(x);
                   return p(x-3);
            end;
        end;
    end;

k(f)(2);
```

It does not appear that the difficulty is with programmming. Subjects who understand the syntax quite well and are able to write fairly complex code, simply cannot deal with the idea that `k` operates on a function and returns, as the result of its operation, a function — as opposed to a number, or a collection of numbers. Given that difficulty, there is little hope that they will realize that the result of the last line is the quantity `f(-1)`.

The effects of such difficulties go beyond the elementary study of functions. Moving up to calculus, it is not surprising that students who cannot handle the ideas in such computer code have trouble understanding differentiation and integration as processes that act on functions and produce functions. Moreover, the whole concept of a differential equation may appear inaccesible to such students.

#### 1.1.3 Processes and Objects together

In many mathematical situations it is necessary to go back and forth in interpretations of a function as a process and as an object. Consider, for example the following problems. We assume that students have studied the concepts of 1-1 functions, addition of functions, and composition of functions.

For each of the following statements, explain why it is always true or give an example for which it does not hold.

1. The sum of two 1-1 functions is again 1-1.
2. The composition of two 1-1 functions is again 1-1.

If you are thinking, for example, of a typical class in discrete mathematics for preservice high school teachers, say in their third year of college, most people who have given such a course would not dream of putting questions like these to their students in an exam. The reaction would be total confusion on the part of most students.

#### 1.1.4 Graphs

I think that many teachers have observed that although students can learn to work with graphs, they do not necessesarily make the connection between a graph and a function. In particular, they cannot say how the function's process is displayed in the graph. Rather, they are restricted to plotting points and a mechanical application of the "vertical line test." They are not even likely to be comfortable about a "horizontal line test".

One thing that adds to students' difficulties here is the fact that although a graph and a function represent essentially the same thing, one representation is static and can live on paper, the blackboard or a computer screen whereas the other is dynamic and lives mainly in the mind of the subject.

An interesting specific example in which the missing connection is manifested has to do with students' "intuition" about tangents to curves. It is generally considered that in teaching the concept of limit, one should rely on the students' natural intuition of the tangent as a line which represents the limiting position of a secant connecting two points as the first point moves towards the second. Most teachers believe that this is a very natural idea. I am not so sure.

I have performed the following "experiment" with several groups of undergraduate students with similar results. I draw the classic picture on the blackboard.

Then I redraw $x_2$ a little closer to $x_1$ and connect with the secant. I repeat this with positions for $x_2$ closer and closer to $x_1$. At all times the students seem comfortable with what is going on and are able to predict what will happen next. Then I stop and ask them to imagine the situation if $x_2$ continued to approach $x_1$. After giving them a moment to contemplate, I ask what will ultimately happen to this secant line.

I find the results quite surprising. About $\frac{1}{3}$ will say that it will be the tangent at the point $P$. This is fine. But another third usually claim that it will be the horizontal line

Figure 1: Secants Approaching the Tangent

through $P$. About half of the rest mention the vertical line through $P$ and the others give uninteresting responses.

The students stick to their answers, even in open-ended interviews. Several point out that the rise is getting smaller and smaller so the line must be settling down to the horizontal. When it is suggested that the run is also disappearing, they respond with comments like, "Oh, I didn't pay any attention to that".

It seems that such students need a little more than appeals to their geometric intuition if they are to understand the notion of a derivative as the slope of the tangent.

A final example related to graphs is an experiment now being performed by one of my undergraduate students. He gives the students a sheet of paper with two columns of numbers which, they are told, represents a table of values of an independent variable and the corresponding values of the dependent variable given by some unknown function. The task for the students is to make a rough sketch of the second derivative of the unknown function.

One intriguing result from students who have completed a year of a traditional calculus course is that they are often able to draw the graph of the first derivative but not the second.

### 1.2 Mathematical Induction

It is not difficult to train students to use mathematical induction to prove that a particular sum of $n$ finite quantities is equal to a given expression in $n$. However, they cannot use this method in situations that are in any way unique or unfamiliar. For instance, they are unable to show that a number of the form $11^{n+2} + 12^{2n+1}$ is divisible by 133, or that a casino with only \$3 and \$5 chips can pay off any dollar amount of \$8 or more.

When faced with induction problems, many students complain that they simply do not know how to get started. Often they have a very strong confusion between proving the original statement of the form $P(n)$ and proving the "corresponding implication", $P(n) \Longrightarrow P(n+1)$. How many math teachers, on pointing out that what must be done is to first assume $P(n)$, hear immediately the complaint that this is circular, that they have assumed what they must prove?

A question that I use to tease out a student's level of understanding of induction is to give them a fairly simple induction problem and get them to give a proof, even with prompting. Then I ask them how they would convince a friend (not mathematically sophisticated) that the statement is true, for say $n = 500$. What I usually find is that the typical college student simply has no clue as to why an induction proof actually gives you some information or allows you to make predictions about the results of calculations. Details of such interviews are given in [6,12].

### 1.3 Quantification

One can give a long list of mathematical concepts which cannot be even stated, much less analyzed without the use of existential and universal quantification. In an investigation performed with Bernard Cornu in Grenoble [4], we asked French students of age 17, to look at the following two statements and, in each case, explain why it is true or give an example in which it is false.

1. For every positive number $a$, there is a positive number $b$ such that $a$ is less than $b$.
2. There is a positive number $b$ such that for every positive number $a$, we have $b$ is less than $a$.

A very high percentage of the students got the first statement correct and even gave a reasonable argument for why it is true. Very few got the second as well. They tended to say that it, also, was true and stuck to their position in interviews. In many cases they justified their response by insisting that the second statement was the same as the first. As with the secants, when the difference in ordering the phrases within the statements was pointed out, they simply claimed that they had paid no attention to that.

Surely students who cannot distinguish betweeen the above two statements are severely limited in the mathematical topics they are capable of understanding.

### 1.4 Groups

Moving from Discrete Mathematics to Abstract Algebra, I can only make some preliminary statements as work in this area is just beginning. It is being done jointly with Uri Leron of the Technion.

Students do not seem to take easily to the idea of an axiom system. They tend to feel that the domain of mathematical calculation is a place for universal and immutable truths. The idea of a set of rules, valid for one discussion (or on one page of the book) but not for others is very strange for many people.

It is not easy to get students thinking about "Group" in general as opposed to a particular group. They can learn to check a particular example for commutativity, or being a subgroup, or being normal. The general idea of such properties, however eludes them. They do not take easily to the idea of defining a property, checking various examples for this property and then trying to prove things about the property.

One particular item of great difficulty is the notion of coset multiplication which is, of course, critical to the idea of quotient groups and Lagrange's theorem. We find students highly resistant to the idea of performing an operation on two cosets and obtaining, as a result, another coset.

# 2 A Theoretical Framework for Explanation

In this section I want to outline a general theory of mathematical knowledge and how it may develop in an individual. The essential point of the theory as far as this paper is concerned is that each individual must construct for her- or himself, in a social context, various coherent collections of mental objects and processes to use in order to make sense out of mathematical problem situations. The theoretical aspect of our investigations consists of trying to determine, relative to individual mathematical topics, the nature of specific objects and processes that must be constructed and the means by which this can be done.

The starting point for each individual investigation of a mathematical concept is the difficulties that subjects have when they try to learn the concept. That is why we began this paper with some examples of student difficulties. The next step is to attempt to use the theory to explain the difficulties. In the following section I will illustrate this for some of the examples described in the previous section. Based on these explanations, we try to design instructional treatments the main thrust of which is to present students with problem situations designed to induce them to make the specific constructions that, according to the explanations, appear to be difficult to make. This will be illustrated in Section 4. It is with these instructional treatments that computers enter the picture.

## 2.1 A paradigm

It is a mistake to think that a theory can be presented without background. Aside from its explanatory power, an important point to consider in evaluating a theory is its source. What is the research paradigm which has led (or is leading) to it?

In the case of the theory we are using in this paper, the paradigm is basically a circle. By that we mean that it is to be repeated over and over, hopefully on higher and higher planes of thought.

It also means that a discussion of it, which must be linear, has to break in at some more or less arbitrary point. Let us begin with the design of instruction. Somehow or other, one decides to do something with a class that relates to the mathematical concept that is to be learned. The instructional approach is implemented and the students are observed. Next, all of the data that comes from the observations are organized and analyzed. The analysis depends on the data. It can consist of organizing responses into categories, counting the number of answers of different types, rating performance, or any kind of reduction to a manageable amount of information that represents at least some aspects of the entire collection of data.

The results of the analysis are then considered in light of the theoretical epistemology — that is, the researchers' current interpretation of what it means to understand this particular topic. Serious consideration is given to a general theory on which the particular investigation is based and also to the researcher's personal understanding of the mathematics in question. All of this is coordinated with the data and the epistemology as understood at the moment. If necessary, the epistemology is revised. It may also happen that the result tends to confirm aspects of the existing epistemology. As time goes on, not only the epistemology of the particular concept under study may be revised, but aspects of the general theory are also occasionally reconsidered.

Now, the researcher's new understanding of what it means to learn this particular topic is used to redesign the instructional treatment and the entire activity is repeated, perhaps at a later time with a different class. The iterations continue as long as desired to hopefully converge on a better understanding of the student's construction of this particular topic and how instruction can help her or him make that construction. In particular, it is also expected that the effect of the instructional treatment on student learning improves as the paradigm is iterated. Ultimately, this is the real test of the theory *and* the paradigm.

### 2.2 The nature of mathematical knowledge

The paradigm just described is very general. One aspect is the general theory we are using. We begin with a brief statement intended to encompa