MATHEMATICAL LITERACY AND ABSTRACTION IN THE 21^{st} CENTURY

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What should be the level of knowledge, understanding, and ability relative to mathematics for the college graduate in the 21^{st} Century? Although this question is important for the specialist in mathematics, I would like, in this Chapter, to consider the non-specialist. I include, of course, students in all sorts of areas that relate strongly to mathematics such as the physical and life sciences, computer science, engineering, management, architecture, linguistics and library science. I include students in areas not generally thought of as having to do with mathematics such as english, history and yes, even cultural studies. What I have to say about mathematical literacy applies to students in all of these areas because, as I will try to show, the need for the kind of literacy I will describe is present for every educated citizen who would like to have a more than superficial understanding of the society in which he or she lives.

Whatever this knowledge, understanding and ability, this literacy, may be, my point is that it must include, as a major component, abstraction. The nature of increasing levels of intellectual sophistication in mathematics consists, to a large extent, of increasing levels of abstraction (Piaget, 1985, pp. 149-150). It follows that if there is to be an increase in mathematical literacy of the general population, then there must begin in the schools and continued in the colleges an increased concern with helping students develop strong and useful relationships with abstraction. I mean primarily abstraction in mathematics, but as I will explain below, the need for abstract thinking goes well beyond any one subject and permeates many human endeavors that will be of growing importance in the new millenium.

What I am proposing for K-12 and post-secondary mathematics education is not just a change of topics, but rather a major departure, an entirely new point of view. To summarize it in a sentence, I propose in this essay that we do not try to make ideas less abstract for our students, but rather help them learn how to deal with abstraction, how to understand it, how to make use of it. Much of this essay is both an argument for this position and at least the beginning of a proposal of how it might be done.

Let me give one illustration of a part of what I am trying to say. Although somewhat negative, it indicates the magnitude of change I am suggesting. Very often students respond (on exams, for example) to a request for a proof of a certain assertion by giving one or more examples in which the assertion is true. Most teachers find it frustratingly hard to get students away from this habit of mind. Often a proof requires thinking in abstract generalitites, whereas the student prefers, indeed insists on, staying with concrete specifics. I suggest that at least a part of this preference is learned from us, the teachers. A student is struggling to understand a concept and asks the teacher for additional explanation. Although he or she may not realize it, the student may be asking for help in developing an abstract understanding. How often, however, does our response begin with "Let me give you an example". In my view this amounts to a suggestion that the student turn from the abstract to the concrete. Of course, we know that asking for a proof is different from asking for an explanation, but the difference is subtle and can be lost on the inexperienced student. Small wonder then that such a student responds to a request for a proof with "Let me give you an example."

Reconsidering the role of examples and focusing on the abstract is very different from how most teachers of mathematics think of their work. I have no illusions about the likelihood of achieving it in our society. But I believe that such a change is very much needed and if we do not achieve at least a large measure of it, our complex society may be in big trouble. So I think it is worthwhile thinking, as I am trying to do in this essay, about the nature of abstraction, its role in our society, the difficulties in learning to think abstractly and an approach to helping students deal with abstraction. Before doing all that in subsequent sections, I will begin, in the next section, by trying to explain in a little more detail, some of the social reasons for abstract thinking.

1 The case for abstraction

There are (at least) two ways in which abstraction is present in our society and plays an important role. One is a large and increasing number of activities engaged in by growing numbers of people and requiring abstract thinking to understand. The other is the highly rapid change in almost all products and services with which we have to deal in our work, our homes and even our play. In this section, I will only list some examples. I must explain (in the next section) what I mean by abstraction before considering (in Section 3) what these examples have to do with abstract thinking.

1.1 Abstraction in our complex society

Following is a list of examples of activities the understanding of which can involve abstract thinking. Many of these can be difficult for many people to understand. For each item on the list I will give a statement that is commonly made followed by some questions this statement raises.

Changing rate of change. "The rate of increase in the national debt is decreasing." Does this mean that the amount the government owes is going up or going down? Why are the contradictory words 'increasing' and 'decreasing' used?

Survey error. "According to the polls, the incumbent leads the challenger by 52% to 48%." The statistical error in this survey is ±3%." What does this tell us? Since the difference between the two is 4% and the error is only 3%, does that mean the incumbent is ahead?

Weather prediction. "The probability of rain tomorrow is 70%." What does this mean? Is it that if tomorrow comes 100 times then about 70 times it will rain? How does this help us decide whether or not to hold the picnic?

Currency exchange rates. "A US dollar is worth 3 German marks." How is this determined? Why does it change everyday? What does it tell us about the relative strength of the US and German economies?

Inflation. "Inflation is making my money worth less." Is inflation universally evil or does it have some advantages as well as disadvantages?

Running a large organization. "Even in the areas in which the law gives the president total authority, he or she has only a limited amount of control over what actually happens." Does this mean that government officials are working against the president and ignoring offical instructions? Why are there limitations? Are there ways in which changes can be made top down?

1.2 Rapid change and abstraction

We are a society of incredibly rapid change. In technology, for example, it could be argued that there has been more change in the last 50 years than in all of recorded history. Some of this change is artificial in the sense that it comes from "planned obselescence" and serves to make a particular economic system function. A great deal of it, however, amounts to solving problems, to enhancing human activity such as the ability to communicate rapidly over huge distances, to produce and manipulate 3-dimensional video images as a design tool, or to store and retrieve information at quantity and speed levels undreamt of even a few decades ago. All of this change increases the complexity and sophistication of everything we deal with from telephones to telecommunication.

Rapid change creates problems for the workforce. Those involved in design, production, maintenance, sales or planning are constantly faced with the need to learn how a new system works. Perhaps the largest change this implies for our educational system (at all levels) is that there are very few areas in which a student can learn in school how to do something that he or she will be required to do after joining the workforce. If a person is going to work in the computer industry, say as a systems analyst, there is very little value in focussing what he or she learns on the syntax of a single, or even of several, programming languages or machine languages. Whatever specifics are learned will be obsolete by the time of graduation, or shortly thereafter. Similarly, someone who wants to work as an auto mechanic will not be well served by an education that consists mainly of learning the details of how large numbers of automobile engines are made, how they work, what can go wrong and how it can be fixed. If the student wants to be working on relatively new cars, again by graduation time, most of those facts will be irrelevant. It may be that some jobs will pay well at first for graduates who are well-versed in present-day circumstances, but advancement and job security will go to those who know how to adapt to new situations.

Our educational system has to move from emphasizing knowledge of what exists today to helping students develop the ability to learn, rapidly and thoroughly, new things, even new conceptions that will exist tomorrow. Understanding general principles is becoming far more important than knowing lots of facts. I will argue in what follows that abstraction is deeply involved here, both in understanding the role that general principles play viz a viz specific facts and in helping students become capable of dealing with rapid change.

2 What is abstraction?

Since this term means so many things to so many people in so many different contexts, it is not reasonable to give a definitive answer that will recieve general acceptance. Rather, I will clarify what I mean in this essay by abstraction and hope that readers will at least agree that abstraction is somewhat like what I am saying.

By abstraction, I am referring to any thinking that tries to "deal with" phenomena to which we do not have access solely through our five senses but rather exist only in our minds and/or our interactions with others. By "deal with" I mean things like making sense out of, constructing (mentally), manipulating, and/or applying one or more aspects of the phenomena.

This notion is different from the commonly held understanding of abstraction as the extraction of common features from a large number of examples. Indeed, one can (and historically this is what happened) create the abstract mathematical notion of *group* by looking at essentially one example, the set of permutations of the roots of a polynomial that leave invariant a certain set of functions. In my opinion, the notion of extraction is a useful, but not very powerful tool for the mental construction of mathematical concepts.

I would also like to disassociate abstraction from the equally common idea of an abstract/concrete dichotomy. For one thing, many people, especially students, often say that a concept is abstract when what is really meant is that they don't understand it. Indeed, concrete is a very relative term, depending not only on the person, but on which aspect of a concept is being discussed. Consider, for example, the function concept. If a function is given by a formula representing some common human activity then it is concrete for most people. On the other hand, the idea of a domain set, a range set and a method (explicit or implicit) for transforming elements in the domain to elements of the range is not at all concrete for many people. But it is for those who see such a triple as an object which itself can be transformed (e.g., by arithmetic operations, by composition, or by placing it as an element of a set.)

Enough of what I don't mean by abstraction. I need to say something about what I do mean.

For me, abstraction in general is the determination in a given situation, which may be a mathematical object, a procedure, or a combination of the two, of what is essential in a component of the situation. In mathematical abstraction, one generally expresses this essence in some systematic manner such as formal language or a set of axioms. Thus, one might consider the integers and focus on the component of addition. The essential aspects might be taken to be: the procedure of adding two integers to get a third; the properties of associativity and commutativity; the fact that if one of the two integers is 0 then the third is the same as the other; and the existence of negative numbers together with the fact that if the two integers are a number and its negative, then the third is 0. This essence is then expressed, using formal language as the set of axioms for a commutative group. Of couse, if one focuses on other aspects of the integers, one can perform different abstractions to obtain, for example, concepts such as semigroup, ring, ideal.

Notice that abstraction, as described here, can be and is, done with respect to a single example. In the case of groups this is what happened historically, although not with the integers but, as I indicated above, with a group associated with a certain subset of the set of permutations of the roots of a polynomial.

This idea of abstraction assumes that a certain situation is present and the individual has some understanding of its meaning. Abstraction then may be considered as a way of focusing (using formalism, for example) on an aspect of the meaning of a situation. But it can also go the other way. One can begin with a formal description and, by analyzing the formal expressions, construct for oneself an understanding and a meaning that was not present before. For more about this notion and its relation to abstraction, see Dubinsky (in review).

I will discuss below, in Section 5, a more specific description of the act of abstraction, as I have described it, in terms of the mental construction of mathematical actions, processes and objects. But first I want to relate this idea of abstraction to some of the examples in our society that I mentioned above. Then I want to say something about the difficulties students have with abstraction.

3 Abstraction in our society

All of the example given in Section 1.1 are (non-mathematical) concepts that arise by means of abstraction. I will illustrate what I mean in terms of two of these examples.

Changing rate of change of the national debt. Understanding this begins with the idea of a national debt, which is a dollar amount, that changes with time. That is, one must imagine, without necessarily knowing anything about how it is determined, that for any given time, the government owes a certain amount of money. One might have a general understanding that given any numerical quantity that varies with time, there is a notion of (instantaneous) rate of change of that quantity. It can be a very intuitive notion of how much the quantity changes per unit of time over a very short period of time, or one can be more sophisticated and think in terms of limits and derivatives.

In any case, one must perform a rather difficult mental act and think of this rate of change as an entity in itself, which also depends on time. That is, for any given time, the national debt is changing at a certain rate. Again, it is not necessary to know what this rate is or even how it is determined.

Now, one can do the same for the rate of change of the national debt as was done with the national debt. One may consider small time intervals and compare the rate of change at the beginning and end of each interval divided by the length of the time interval. Again, either staying at an intuitive level or using the concept of limit, one arrives at the notion of *rate of change of the rate of change of the national debt.*

If this concept is well understood, there is no difficulty in understanding that the national debt may be increasing (rate of change) but at a decreasing rate (rate of change of rate of change.)

Currency exchange rates. There are (at least) two issues involved in understanding the nature of exchange rates. How does a bank determine each day rates at which it will exchange currency — for tourists for example? What are the underlying forces that determine the facts on which banks make these determinations — so that, for example, reasonable predictions might be made?

The first question has a very simple answer. There exist a number of centers where people buy and sell currencies. These are called Foreign Exchange Markets. Individuals offer to buy or sell various amounts of currencies of one country for currencies of another currency. These offers are accepted or not and so the prices go up or down. A bank determines the exchange rates for its customers on a given day by noting the prices that actually occur in these markets.

The second is more complicated and has several different answers. One answer used by many prognosticators is purchasing power. That is, the rate of currency of country A with respect to country B is the ratio of the price in country A of a certain product to the price in country B of the same product. Of course there are questions to resolve such as which products, what kind of averages to take, etc. Each prognosticator resolves these questions for her or himself and makes the prediction.

Understanding currency exchange rates requires a synthesis of these two issues, a coordination of the actual practice of a large number of people with an imagined set of purchases of various goods and services.

At this point, the relationship between examples from our society like these and mathematical abstraction as described in Section 2 is vague at best. After considering, in the next section, some of the difficulties students have with mathematical absraction, I will describe very briefly, in Section 5, one theory of learning abstract abstract mathematical concepts and relate some of the details with these non-mathematical examples.

4 Difficulties in learning to deal with abstraction

According to Hazzan (1995), most students' misconceptions can be attributed to their tendency to work at lower levels of abstraction than are necessary in order to understand a particular concept. This point has been made by a number of authors over a number of years (see, for example, Leron (1987), Wilensky (1991), Sfard (1992), Dubinsky et al (1994), Noss and Hoyles (1996), Staub and Stern (1997).) These and many other papers report the fact of student dificulties. It is probably the case that abstraction is the most serious barrier to student success with mathematics and, I would argue, a number of sophisticated concepts in our culture such as the examples I have listed.

Obviously, we will be very limited in our progress towards increasing the level of sophistication in our students' thinking unless we develop methods of overcoming their difficulties with abstraction. There is very little in the literature reporting success with any general methods for doing this. Perhaps this is to be expected. What success has been achieved is in methods for helping students understand specific concepts.

Generality is achieved through general approaches for dealing with specific concepts. I will describe one such approach in the next section.

5 One approach to helping students learn to deal with abstraction

At the end of Section 2, I referred briefly to ways in which formalism and mental constructions can interact in an individual's development of her or his understaning of an abstract mathematical concepts. Key to this interaction is a general theory of learning abstract mathematical concepts which describes certain categories of mental constructions which can be applied to a specific concept. I will describe this general theory and the categories of construction with examples taken from the domain of mathematics. Then I will indicate how some of the (non-mathematical) examples from Section 1.1 can also be seen in terms of these mental constructions. It is this connection that causes me to suggest that if students can learn to think in abstract terms in a mathematical context, then that high-level kind of thinking can also be applied to non-mathematical situations.

Finally, I will say a little about the instructional treatments that have been designed under the influence of this theory and point to the results of using it in college courses over the last dozen or so years.

5.1 APOS Theory

APOS theory is a constructivist theory of how learning a mathematical concept might take place. According to this theory, learning mathematics consists of making certain kinds of mental constructs in response to mathematical problem situations. The general theory describes these constructions in terms of what are called actions, processes, objects and schemas. An APOS analysis of a particular concept consists of describing, for that particular mathematical concepts, the specific constructions and their relations.

The main building blocks in these constructions are actions, processes and objects.

Actions.

Understanding a mathematical concept begins with performing *actions* on (mental or physical) objects that transform them into new objects or the same objects with new attributes. In order for an individudal to perform an action, he or she must have in her or his consciousness a specific list of instructions and the action must actually be performed down to the last detail. Whether the recipe for an action is given by another person, written down or recalled from memory, the individual perceives that the performance of an action is externally driven.

The clearest example in mathematics is the concept of *function*. With an action conception of function, the individual requires an explicit formula defining the function and about all that he or she can think about with resect to functions is the subtitution of a number, or at best another formula for the variable in the given formula and simplify.

For a slightly more sophisticated example, if the concept in question were *cosets*, learning would begin with an *action* conception consisting of forming cosets which can be described by listing their elements, such as the cosets of the subgroup of multiples of 4 in the group of integers mod 24 with addition mod 24 as the operation. Calculations could be made of specific cosets and properties such as the number of elements in a coset, the number of cosets of given subgroup, and disjointness could be observed. At this level of understanding, the student would not be able to consider larger and more complicated groups and a binary operation on the set of cosets of a subgroup would be hard to understand. The idea of a quotient group would be essentially inaccessible.

Processes.

A higher level of understanding would be a *process* conception in which the individual interiorizes the actions. He or she no longer requires explicit listings and calculations to perform operations but can imagine them or run through them mentally. The transformations involved are seen as internally driven and the individual can imagine a process being reversed or two processes being combined in some manner.

Thus, for the function concept, a process conception would be an ability to understand a function as an input/output operation. An object goes in, something is done to it, and a new object comes out. A function does not have to be defined by a formula. Functions can be composed and inverted.

For the case of cosets, given a group G, a subgroup H, and an element $x \in G$, the individual can think of running through the elements of H and forming the group product of each with x (on the left, say). Not only does this allow the individual to work with more complicated groups, such as the group of permutations of n objects, but he or she can also think about other processes such as figuring out cardinalities and forming the set of all products of two group elements, the first from one specific coset and the second from another.

Objects

Given a process, there can arise situations in which it is necessary to apply actions or processes to it. In order to do this, the individual must encapsulate the process to become an *object*. This is for the purpose of thinking about performing a transformation. Often, in order to actually perform a transformation on an object, it is necessary to de-encapulate the object back to the process from which it came.

Consider, for example, the operation of composition of functions. One thinks about this as taking two functions, combining them in some way and thereby obtain a new function. From this point of view, all three functions are considered as objects. But to actually perform the transformation, one must de-encapsulate the two input functions back to their processes, coordinate these two processes by imagining the output of one being sent to the input of the other, and finally, encapsulating the resulting process to an object.

In moving towards thinking about properties of cosets and performing actions on them, the individual must encapsulate the process understanding and develop an *object* conception of cosets. Then it becomes possible to consider theorems concerning the cardinality of every coset of a particular subgroup, the number of cosets and their disjointness.

At this point, major ideas such as Lagrange's theorem become accessible to the student and he or she can think of constructing a binary operation on the set of cosets of a subgroup, determining its properties and thereby getting to the ideas of normality and quotient groups. In working with these concepts, it is important for the individual to pass easily from the object back to the process from which it came and then return to the object as needed to work with particular situations. Thus, having constructed (mentally) the set of cosets of a subgroup, the individual can think about a binary operation which takes two cosets and produces another coset. Such thoughts require an object conception of cosets. But to actually construct the binary operation (as the set of all products of two elements, one from each coset) the individual must return to the process conception. In many mathematical activities, it is necessary to go back and forth between process and object conceptions of a mathematical entity.

Schemas.

All of these interpretations of a mathematical concept, together with properties and relationships that the individual understands about the concept are organized in what we call the individual's *schema* for the concept. A schema is a collection of actions, processes, objects and other schemas, together with their relationships, that the individual understands in connection with the concept. This collection will be coherent in the sense that the individual will have some means (explicit or implicit), perhaps the formal definition, of determining, for any phenomenon encountered, what relationship it has to her or his conception of this particular concept.

Finally, a schema can be thematized to become an object in much the same way that a process is encapsulated to an object.

5.2 Examples of APOS Theory from society

My point does not requre showing that situations in society can be fully described and analyzed in terms of actions, processes and object. All that is necessary is to indicate how the mental constructs have some relationship to aspects of an example. This can then be taken as support for my contention that working with abstraction in mathematics can help an individual understand these complex examples that we must deal with in our lives. At the very least, APOS Theory can help us understand why many people have difficulty in understanding some aspects of our complex society. For example, if we find that encapsulation of processes in mathematics is a mental operation that is very hard for students, takes a long time, and even requires sophisticated pedagogical interventions, and if we can see the need for encapsulation in a concept such as changing rate of change of the national debt, for example, then not only does this help us understand why so many people misunderstand this politico-economic concept, but it also can point a way to helping individuals overcome the obstacle.

I will consider two examples: changing rate of change and currency exchange rates. I refer the reader to Section 3 for a discussion of these concepts.

Changing rate of change

The connection here is quite clear. The change in, say, the national debt is a process in which the amount of the debt at one time period is "transformed" to the amount of debt at another time. Conceptually, this change can be thought of as diferences over a unit time period, such as a day or a year or a presidential term of office. It can also be considered as an instantaneous rate. In either case, the process must be encapsulated to an object. Thus we have the rate of change as an object which can also change in time, giving a "higher order" rate of change. This is the concept of a ate of change of a rate of change.

The fact that this concept fits so neatly with the mental constructions that can be used to construct mathematical concepts is due, of course, to the fact that it is essentially identical to the mathematical concept of the second derivative of a function.

Currency exchange rates

The connection between APOS Theory and currency exchange rates is a little less complete than in the case of changing rates of change. The main point is that in thinking about how currency exchange rates are determined, one has to imagine a large set of transactions. Indeed what is involved is the purchase of every product made by two countries and the cost of each sale in each of the two countries. Of course no more than a small number of possible transactions are actually made, but all that are possible must be considered as potential in deciding on the exchange rates. Thus the transactions are processes and not actions. It seems likely that an individual who requires an explicit formula to understand and work with a function is going to have some difficulty in thinking about *potential* transactions and so with international exchange rates.

5.3 Instructional treatment

Our pedagogical approach is part of an overall framework for research and curriculum development in undergraduate mathematics. I will sketch this framework briefly and refer the reader to Asiala et al (1996) for more details. The framework has three components: theoretical analysis, instructional treatment, and data collection/analysis, organized as indicated in Figure 1.

In Figure 1, the arrow going from the theory to the instruction represents the specific mental constructions which the theoretical analysis proposes to the design of instruction. This design focuses on getting the students to make these mental constructions with knowledge of the mathematics involved as an expected by-product.

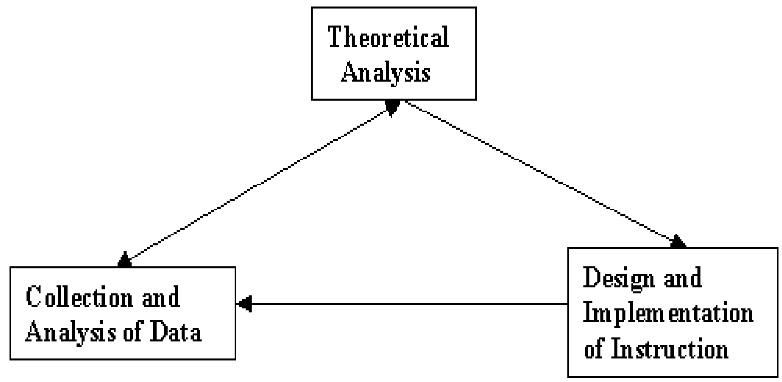

Figure 1

Several specific pedagogical strategies are used to help the students make the desired mental constructions. The most important of these are having students write computer code to implement mathematical concepts, and cooperative learning. In addition, there is an attempt to have the students engage in active learning, to figure things out on their own rather than rely on the instructor for explanations and information. This results in less lecturing than in a traditional class. Inputs from the instructor are designed to help students make the mental constructions. When the students have had an ample opportunity to make these constructions, the instructor may explain various concepts and methods to a small group of the students or to the whole class. These explanations are, again, based on the mental constructions the students are expected to have made.

The instructional treatment is organized in what is called the *ACE Teaching Cycle* of Activities to be done on the computer, Classroom discussions, and Exercises to be done with pencil and paper. A typical class will meet one or two days each in a computer lab and two or three days in a classroom.

In the computer lab, the students write computer code and programs and use these constructs to perform mathematical tasks. The computer activities are specifically designed with the mental constructions in mind. For example, to help the students construct an action understanding of cosets, they might be asked to write code such as:

```
G := {0..19};
op := |x,y -> (x + y) mod 20;
H := {0,4,8,12,16};
K := {3,7,11,15,19};
```

where the first two lines construct the additive group of integers mod 20, the next line forms a certain subgroup H and the last line gives the coset $K = 3 + H$.

For a process conception, the student might write,

```
G := {[a,b,c,d] :  a,b,c,d in {1..4} | #{a,b,c,d} = 4};
op := |p,q -> [p(q(i)) :  i in [1..4]]|;
H := {[1,2,3,4], [2,1,4,3], [3,4,1,2], [4,3,2,1]};
K := {[2,3,1,4] .op p :  p in G};
{p .op q :  p in H, q in K};
```

where this time the code constructs the group of permutations of four numbers with composition as the operation, the subgroup H which is isomorphic to the Klein 4-group, a coset K of H and the coset product HK.

Finally, to help the students construct an object conception of cosets, they might be asked to write a program that will accept any two cosets and return their product. It would look like,

```
CP := func(C1, C2);
            return {x .op y | x in C1, y in C2};
        end;
```

where it is assumed that a group operation `op` has been defined. This program can then be applied by the student to specific examples and used to investigate properties of the coset product, such as commutativity.

For more information on this way of using computers, see Dubinsky (1995).

In the classroom sessions, the students are given specific mathematical tasks to perform, based on the mental constructions they have made in the computer lab. For example, they might be asked to make a general statement about the number of elements in a coset or the intersection of two cosets, based on the examples they have worked with on the computer. Then the class could move to a proof of various properties that the students have observed empirically. In addition to working on these tasks, from time to time the students will listen to explanations (or brief lectures) by the instructor, who must decide when to let the students try to figure out something on their own and when they are ready to hear an explanation. This interplay between discovery and explanation is where the teacher has the greatest opportunity to control the pace of the course and apply her or his pedagogical creativity.

Finally, exercises are assigned to do as homework. These are fairly traditional drill and practice as well as problems that require deeper thought. It is important to note that, unlike traditional instruction, the number of illustrative examples is minimized until the students have had ample opportunity to construct understandings of the mathematics involved. The reinforcement that comes from practice is an important part of learning, but it should not take place until the possibility of reinforcing misconceptions is reduced as much as possible.

All of the students' work in the course in the computer lab, in class, on the exercises and even some of the examinations, is done in cooperative groups which are established at the beginning of the course and not changed thereafter.

5.4 Data collection and results

A number of studies were made of courses conducted according to this framework and a very large amount of data was collected. Descriptions of methodologies and summaries of results are given in Weller et al (in review). There is not sufficient space here for any details, but perhaps the reader will be interested in the final paragraph of that report.

> Our results seem to point to the success of this theoretically-based approach as a valid tool by which students learn advanced mathematical concepts. However, the results reported in this study were culled from a number of research papers whose primary methodology was qualitative and whose aim was to reveal the nature of students' understanding rather than to compare students' performances statistically. A statistically sound, quantitative, independent comparison study may build upon these results. The authors

call for other researchers in collegiate mathematics education to conduct further comparative evaluations of the performance of students who have completed APOS-based courses to those who have received traditionally-based instruction. Further, it is hoped that the results presented in this paper will motivate teachers of mathematics to learn more about these approaches and to incorporate these ideas into their classrooms.

6 Conclusion

I have tried to reflect in this chapter on the nature of abstraction in mathematics and its relation to phenomena in our society. I do not know of any data to back up my opinion, but I hope that my arguments tend to make it plausible that learning to think abstractly in mathematics could help individuals understand complicated ideas in our complex society. Given the difficulty that anyone has in learning to deal with abstraction, these would be idle musings if they did not include something about new pedagogical strategies to help students overcome these difficulties. I have tried to show, and point to corroborative data in the literature, that a particular approach, based on the APOS theory, that has students writing computer programs, working in cooperative groups, and actively engaging in mathematical tasks, can have a certain amount of success. Thus, in addition to arguing that college mathematics students should learn how to think abstractly, I am making a definite and detailed suggestion of how that might come about.

If my ideas about the role of abstraction in post secondary education have any merit, and if they were to be adopted to any extent in colleges and universities, then this would have important implications for K-12 education. We are reaching the point in our society where almost everyone who graduates from high school attends college. If college students are going to be asked to learn how to deal with abstraction, then there should be some preparation for that experience in their pre-college studies. It is not clear to me exactly what should be done at the K-12 levels to prepare students for abstraction and I, for one, must leave this question to those whose work has a pre-college focus. At the very least, it is clear to me that a focus on abstraction at the post secondary level would imply that pre-college education should not only emphasize how to do things in mathematics but also raise the question of why certain procedures give certain results. Students should come to college knowing how to perform a large number of procedures in mathematics and, if they do not know why they work, they should at least feel a need to answer such a question.

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