UNDERGRADUATE MATHEMATICS — NOW AND THEN

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1 Purpose of Study

There are indications that the mathematical profession in the United States is on the threshold of an intense period of reform and renewal of undergraduate mathematics education. There is no question that many people are calling for such a movement. Reports such as (see References at the end for bibliographical information): A Nation At Risk (National Commission on Excellense in Education, 1983), Renewing U.S. Mathematics (National Research Council, 1984, 1990), Toward a Lean and Lively Calculus (Douglas, 1986), Calculus for a New Century (Steen, 1988), Everybody Counts (National Research Council, 1989), A Challenge of Numbers (Madison & Hart, 1990), A Call for Change (Committee on the Mathematical Education of Teachers, 1991), and Moving Beyond Myths (National Research Council, 1991) tell us that undergraduate mathematics is in trouble. For far too many students, the main thing they get from courses in Calculus, Differential Equations, Linear and Abstract Algebra, is a negative attitude towards mathematics. They express this attitude in many ways, for example by reducing or eliminating any contact with our subject. Those that stay, according to these reports, do not appear to learn very much. Ed Moise put it quite succinctly a few years ago when he wrote,

> For the overwhelming majority of our students, the calculus is not a body of knowledge, but a repertoire of imitative behavior patterns. (Moise, 1984)

If we are going to reconsider and perhaps revise the content, teaching methodology and general characteristics of undergraduate mathematics, then it behooves us to begin with a full understanding of the nature of the presently existing program and its history, at least over the last few decades. It is not just that we want to know what happened in order to avoid repetition of errors. It is important to understand what has changed and why. It is also important to assess whether the changes have succeeded or failed as well as whether what has remained unchanged is succeeding or failing.

Perhaps most important for the present work is the question of "going back to the good old days." Have we lost our way? Did we once know how to teach undergraduate mathematics, but in recent years, have we turned away from the "path of righteousness" and allowed the educational component of our profession to deteriorate? If so, a large part of the change that may be coming should consist of studying the old ways and, perhaps with some modernization, revive them.

But there is another point of view. Maybe undergraduate mathematics education always was in trouble. Maybe we never knew how to teach undergraduate mathematics well and the only reason that so many first class mathematicians emerged from our system was that these were all mathematically talented individuals. Maybe a lot of students were, indeed, turned off by our courses, but enough remained to satisfy the needs of the times. Maybe it is a case of both “garbage in, garbage out” and “quality in, quality out” with very little contribution by the system. If so, we may ask if the number of students remaining in mathematics courses continues to satisfy the needs of the times.

A main purpose of our study was to consider these two alternatives: have we only lost our way or were we always in the wilderness? We wanted to look at the development of undergraduate mathematics, at least in the first two years of the undergraduate curriculum, to understand something about how it has or has not changed over four decades, from the fifties through the eighties. We hoped that we might be able to decide just how much the reform movement, if there is to be one, should be looking to the past as it retools for the future.

With the assistance of a grant from the Exxon Foundation, we set out to do a retrospective study of the level of mathematical sophistication in elementary college mathematics courses over a period of four decades. How did it evolve in time? We also wanted to look at teaching methods, the use of technology and the overall profile of the courses such as class size and the nature of the student population in these courses.

As will be clear from this report, we were much too ambitious — at least for the resources available and our own skills and imaginations. Beginning with the idea of a comprehensive study of many courses in many schools over 40 years, we found ourselves constantly cutting down. What we have is a relatively informal sample of a few courses — mainly calculus — at 53 schools. We do cover the full forty years, but really only for calculus at a reasonable cross-section of American and Canadian colleges and universities.

Nevertheless, some patterns do emerge. From these we offer some conclusions which should really be taken only as conjectures about what might be true. What we found in our very restricted domain is not entirely what we expected. We feel that certain canons of conventional wisdom do not appear to hold for the sample we looked at. Thus some interesting questions can be raised. At the very least, we conclude that there may be more than one way to think about the problems of undergraduate mathematics education and their solutions.

2 Methodology and Data Analysis

2.1 Data Collection

It was clear from the beginning that getting data on undergraduate mathematics education from more than just the last few years would be quite difficult. Few academics or departments keep data on activities related to teaching for more than the recent past and, even when such data might be available, it was unclear whether or not it would illuminate such matters as how mathematics was taught or what topics were emphasized.

Our first thought was to craft carefully a questionnaire which would enable respondees (department chairs, elderly professors etc.) to fill in data where it was available. We did design such a questionnaire and sent it to half a dozen colleagues for comments. The uniform response was that we were asking for much too detailed data and that the result would be that almost no one would take us seriously.

Suitably chastened, we decided instead to request data in a much less specific manner. So first, in June 1989, we wrote to about 350 departments of mathematics at four- and two-year colleges in the United States and Canada describing what we hoped to accomplish and asking for volunteers to cooperate with us. We told them what we would be asking for in general but gave few specifics. From this mailing we had 151 postcards returned with a positive response.

Then in September 1989 we wrote to all the respondees telling them what we hoped they would send us. This letter is reproduced in Appendix 1.

In response we received a considerable amount of data from 53 departments which are listed in Appendix 2. We think that the colleges and universities represented form a fairly balanced sample from the perspectives of geography, public and private institutions, research universities and mainly teaching institutions. To a considerable degree, however, as we shall describe in the next section, the data we got was disappointing. It was not that it did not conform to our preconceived biases but rather that it was so amorphous and fragmentary that it soon became clear that we would not be able to draw any definitive conclusions from it. Therefore, in an attempt to gather some additional data, in June 1990 we sent the questionnaire reproduced in Appendix 3 to 15 leading American mathematicians. These responses are also described in the next section.

2.2 Description of Responses

From the September 1989 mailing we received about three shelf-feet of data. Indeed, many of our colleagues must have worked quite hard to gather all this data and for this we are most grateful. However, as will probably not be a surprise to any reader of this report, there was little that could be called definitive about the teaching of undergraduate mathematics in the U. S. and Canada over the past few decades. Partly this was because the types of data we received (see below) just did not address the question of change very profoundly and because - and this is one of our significant conclusions - there does not seem to have been a lot of change over this period. (In a sense this is an important conclusion in itself; we return to this point at various places below, particularly in Section 4.1.)

Here is a listing of the kinds of responses we received:

- Most respondents sent us a narrative overview of mathematics at their institutions. This typically included some statistical data about the institution itself and mathematics teaching at it but there was also often some historical narrative about the development of mathematics at the college or university.
- We also received a considerable number of course syllabuses, mainly more recent ones but quite a few going back 20 or 30 years or even more.
- We received lots of old exams and these were among the most interesting data we got. It is not surprising that so many departments keep exams around for a long time, not least because academics like to repeat questions from exams old enough so that they are unlikely to be in dormitory or fraternity files. (This is particularly so if, in fact, it is the case that a course does not change much over the years.)
- Some respondents sent us lists of textbooks used now and in the past but we got relatively little information of this kind, in part, we suppose, because records are seldom kept of textbooks used in courses.
- Although over 20 respondents dealt with the question of the level of sophistication of courses, almost all said that there had been no change, a conclusion for which there is further support below. The reason there were so few responses to this question were, no doubt, that there is just not enough corporate memory in many departments to address this point.

We are sure that almost all our respondents tried hard to give us what we wanted. The inability of many to succeed was certainly due much more to the unavailability of data than to any any lack of assiduousness.

We received 8 responses to our short questionnaire to leading mathematicians. The responses were thoughtful but contained no significant patterns. Even in response to the query about the preparation for calculus in high school mathematics, there was no unanimity although most respondents did agree that the preparation is poorer today than in the past. A couple of the respondents criticized our questions as being too leading. And one thought that we must be "guilt-ridden" because we "assumed that since students are learning less, we must be teaching them more poorly". We shall refer to some of the other responses later in this report.

2.3 Analyses of the Data

Although we had asked our respondents to send us data on Calculus, Linear Algebra, Differential Equations, Abstract Algebra and Discrete Mathematics courses, almost all the data we received was, not very surprisingly, about Calculus so that we focused our analysis entirely on the three-semester calculus sequence. We did two main analyses:

1. course syllabuses with respect to coverage of topics in the calculus over time;
2. examinations from the various schools in terms of number of occurences of problems in certain categories.

3 Results

3.1 Introduction

As we have indicated above, any expectation we might have had of making a definitive determination of what undergraduate mathematics education was, what it is, and how we got from there to here was unrealistic. The data we have is an amorphous, incomplete mixture of numbers, summaries, anecdotes and opinions. The fault does not lie with the mathematics departments who received our queries. We were surprised and gratified at their willingness to make considerable efforts to supply us with data. The problem was to find a feasible middle ground between trying to collect and analyze a really representative sample of data (which would have filled several rooms of file cabinets) and a set of information that was too sparse to permit any meaningful conclusions.

In retrospect, it now appears that although we collected more data than we wished to deal with, we obtained too little of the kind of material that would be most useful. For example, the categories in which information is available differ considerably with respect to schools. Thus, although we received a large amount of information from each of about 50 schools, there is no category about which we have complete data from more than about 15 schools and in some cases, the number is as small as two or three.

Nevertheless, we do feel that there is something to be learned from our data. Therefore we present, in this section, summaries of data in some of the categories that we considered. We will focus on Calculus, about which we have the most data and which represents the overwhelming majority of mathematics studied in the first two years of college. We have some interesting data to present about the level of sophistication of this course over the last 40 years. We also have information about the use of technology and material, mainly anecdotal, regarding teaching methodologies, the size and background of the student population in these courses, and some other details such as patterns of grading.

3.2 Level of sophistication of Calculus

The area in which our data is least dubious is the extent to which the level of sophistication of the calculus course has changed over the last forty years. Indeed, the incompleteness of our data on level of sophistication is, to some extent, counteracted by the fact that what we do have seems to be fairly consistent.

We collected three kinds of data on level of sophistication: final exams, course content, and comments from our respondents.

3.2.1 Final exams

We recieved copies of final exams for three semesters of Calculus from 19 schools covering the period 1945-1990. We divided the time periods into five intervals: 1945-60, 1961-70, 1971-79, 1980-85, and 1986-90. We also categorized the questions as follows.

A: Symbol manipulation and calculation. This category includes: A_1 — completely trivial calculations; A_2 — routine calculations which, although not trivial, were the sort of things on which students probably received a great deal of drill and practice; A_3 — complex problems in the sense of requiring a long sequence of routine and/or trivial manipulations together with, possibly, a step or two of fairly difficult calculation; and A_4 — calculations that required thought.

B: Definition, formula, or theorem. This category includes: B_1 — repeat from memory; and B_2 — remember and manipulate to some extent.

C: Word problems. Although we originally had two subcategories, routine and non-routine, we found no examples of non-routine word problems on final exams.

D: Higher level. Here there were two subcategories: D_1 — apply a concept in a way that appears to require some thought; and D_2 — prove something. Originally, we thought of other categories in which a theorem (as opposed to a concept) was applied in a way that required some thought and a question that appeared to require that a proof was understood. However, we omit these categories because we did not find any examples of them in the exam questions.

Examples of questions that were assigned to these categories are given in Appendix 4.

Unfortunately, not every school gave us exams for every time period. Indeed the largest response was for the first semester of Calculus in the period 1986-90 (12 schools) and the smallest was the second semester of Calculus in the period 1945-1960 (2 schools). The total number of questions on all the exams considered was 495. The following three tables give, for each of the three semesters of Calculus, percentages of those questions which we found in each of the categories described above.

Time period	A_1	A_2	A_3	A_4	B_1	B_2	C	D_1	D_2
1945-1960	5	60	10	5	0	0	0	5	10
1961-1970	5	60	5	0	10	0	10	10	10
1971-1979	10	60	5	0	10	0	10	5	0
1980-1985	5	55	5	0	5	5	15	10	0
1986-1990	5	60	5	0	5	0	10	5	0

Calculus I

Time period	A_1	A_2	A_3	A_4	B_1	B_2	C	D_1	D_2
1945-1960	5	60	10	0	10	0	0	0	5
1961-1970	5	80	10	0	5	5	0	5	0
1971-1979	5	70	10	0	5	5	5	0	0
1980-1985	0	75	10	0	5	0	5	10	0
1986-1990	5	65	10	0	5	0	5	5	0

Calculus II

Time period	A_1	A_2	A_3	A_4	B_1	B_2	C	D_1	D_2
1961-1970	0	80	10	0	0	0	0	5	5
1971-1979	10	65	5	0	5	5	5	5	5
1980-1985	0	75	0	0	0	10	0	0	5
1986-1990	0	70	15	0	0	5	5	5	5

Calculus III

In closing this section we should caution against concluding too much from questions on final exams. One of our respondents mentioned that quizzes had more theoretical material than finals. It is not clear how widespread this practice was, nor how important for the student's experience was the quiz as opposed to the final exam.

3.2.2 Course Content

We received copies of syllabi covering the three semesters of calculus for the period 1950-1989 from 25 schools. Again, the data in the returns is sparse. After dividing the time into four year periods, we found that the largest response was in the time period 1986-89 for the first semester of Calculus (13 schools) and the smallest was the period 1954-57 for the third semester (2 schools).

There is an additional difficulty. It is not clear that a topic omitted from a course description is not included in the course (or vice-versa). For example, in only one of the 10 time periods (1954-57) for the first semester of Calculus, did every school mention differentiation. It is not clear what this means. In 1958-61, one out of five schools listed differentiation in Calculus 1 and four out of six listed it in Calculus 2. In some cases this may mean that integration was done before differentiation and the latter did not occur until the second semester. But in other cases, it could mean that the authors of a description of "Differential and Integral Calculus" may not have felt it necessary to mention differentiation explicitly.

Be that as it may, the following items taken from this data might suggest some interesting phenomena.

1. The mention of limits in the first semester seems to increase over the years. In the period from 1950 to 1965 it is only mentioned 5 times out of a possible 18, while in the period 1978 to 1989 it is mentioned 19 times out of 24. The figures for continuity are even sharper, 2 out of 18 and 17 out of 24.

2. The topic of functions appears to have been included in the first semester of Calculus at most schools during all time periods, except 1950-53.

3. The appearance of applications in both the first and second semester course seems to be fairly constant, beginning in the first semester with three out of four schools in 1950-53, having some drops in 1962-65 and 1974-77, but finishing at 8 out of 13 in 1986-89. A similar pattern is seen in the second semester.

4. There is a noticeable decline in the mention of analytic geometry. In the period 1954 to 1965 it was mentioned at 7 out of 14 schools, but from 1978 to 1989 in only 7 out of 24 cases.

5. The topic of techniques of integration in Calculus 2 did not appear very often until the period 1962-65 and from then it is mentioned consistently by almost all schools.

6. It is possible that there is some movement of the topic of infinite series from the third semester to the second semester. In the period 1958 to 1965, two out of 10 schools mentioned it in the second semester and four out of six in the third semester. But from 1978 to 1989, 13 out of 24 schools listed it in the second semester and 7 out of 17 mentioned it in the third.

7. Advanced topics of Mathematical Physics such as Green's and Stokes' Theorems are only sporadically mentioned throughout all time periods.

3.2.3 Comments

We asked our respondents to comment on the question of level of sophistication, from their personal point of view, and also to ask colleagues for their impressions. Most of them did. One must be careful about the reliability of individual perceptions, which may depend on many factors and can be based on different standards. For example, on the question of quality of students, an individual at one school said,

> "...I think that our students right now are better than those...6 years ago and my teaching reflects this."

Another individual at the same school said,

> "I also do not feel that the students are as well prepared for the course as they once were. I find myself reviewing algebra a great deal."

Nevertheless, as with all of our data, the strong leaning in a particular direction might offset the lack of reliability.

In the case of individual comments on the level of sophistication over the last 20-40 years, the leaning is definitely in the direction of "no change". Of 27 schools which commented explicitly on this point, 18 said that things were pretty much the same today as in the past and 6 were of the opinion that the level had fluctuated strongly. Only 3 expressed the opinion that the level of sophistication had declined. Several schools mentioned that there had been a brief period in the middle 1960's during which the level of sophistication of calculus increased strongly (for example, "epsilon-delta methods" were discussed), but this was very temporary. We did not take such a comment alone as indicating fluctuation.

Here are typical comments on the lack of change from two very different schools, one a famous university and the other a large private school.

> "The mainline first year calculus course has stayed fairly stable at least from the 50's to the present."

> "...we do what everybody else does and has done for the last 25 (50?) years."

Here is a typical comment from a school which reported strong fluctuation.

> "...the level of mathematical rigor and the balance of conceptual and computational material has bounced around wildly as the faculty have changed their minds about the correct way to present the subject."

And a few schools made statements such as the following straightforward comment.

> "In general, the amount of rigor and theorem proving required of students has steadily declined...in the calculus sequence."

3.3 Teaching methodology

The responses that we got did not indicate very much in the way of non-traditional teaching methods. Schools in which class sizes were small invariably reported use of a traditional (this word was often used explicitly) presentation via lecture followed by some sort of interaction such as discussion or practice with exercises. Schools in which class sizes were large reported that they used the lecture/recitation method.

There was only an occasional mention of experimentation with some sections using alternative pedagogical approaches.

3.4 Use of technology

Earlier surveys have indicated that use of computers in college mathematics courses was very much the exception rather than the rule. For example, Albers et al (1987), reporting on a 1985 survey, estimated that only 7% of calculus courses, 13% of differential equations and linear algebra courses and 18% of discrete mathematics courses required some computer use. More recently, a 1990 survey by the Conference Board of the Mathematical Sciences found that computer assignments were required in fewer than 10% of all calculus classes (Mathematical Association of America, 1992). Our data does not suggest that much has changed.

Over half of our respondents either ignored the question about technology (see Appendix 1) or indicated that a chalkboard and perhaps an overhead projector were the only "technology" used in lower division mathematics teaching. Indeed, the higher quality the university, the more this seemed to be the case. "The level of technology is largely confined to the chalkboard", said a respondent from a university famous for its use of computers. "Blackboards still represent the major technology", said our respondent from a famous university who also said, "Students in calculus classes are encouraged but not strictly required to have and use hand-held calculators". A number of respondents mentioned calculators in this context, one noting that "calculators are everywhere". We suspect, although it was not explicit, that "everywhere" often means in the remedial courses which are such a large part of the teaching load in many mathematics departments.

We infer that "calculators" always means scientific calculators at most. Only a few mentioned symbolic calculators (e.g. the HP 28-S) and no one discussed the possible effect of such devices on the content and presentation of calculus.

Still there is a hint in our data that we may be on the threshold of a change. The NSF calculus program has a lot of people thinking about the use of symbolic mathematical systems (computer algebra systems) and various places are starting to use them in teaching calculus. Developments at schools designing experimental calculus courses have started to spread to other places as mentioned by one of our responding schools. It is much too early to say how this will come out because there is great inertia in the teaching of calculus and, it would seem, little interest among faculty to use computers in or out of the classroom. Among some faculty there is still the memory of the CRICISAM calculus in the early 70s which proposed computer usage in calculus before most places had facilities to support it and which, therefore, did not result in lasting changes.

There is some recognition that computers have a role in the mathematics classroom. "Our classrooms need to be equipped to display computer output", was a comment from a college that is already heavily computer-oriented in its mathematics teaching and wants to develop laboratories for the teaching of most of the lower division courses.

Most of the computer use is not in the classroom but in laboratories. "Now we do a computer assignment in most classes" means probably that students are given occasional assignments which require them to go to a departmental (or college) laboratory.

Finally, a word about technology and testing, a subject which was mentioned by only one of our respondents who said that while some faculty encouraged their use on tests, others "still feel that calculators should not be used on examinations". But if "calculators" here means four-function calculators or scientific calculators, what is being tested on calculus tests which is better tested without such calculators than with them? In any case, how far away are we from the time when symbolic calculators will be routinely allowed on calculus tests?

3.5 Miscellaneous

3.5.1 Student Preparation

Where it was remarked on at all, there was general agreement among our respondents that the mathematical preparation of entering freshmen has declined. As one person noted, whereas once most students entered college with 3 or 4 years of standard high school preparatory mathematics, now all you can say is that they "have had 3 or 4 years of something". On the other hand it is worth keeping in mind that 40 years ago, even 30 years ago calculus was often or generally a sophomore course so that high school preparation had a different implication. Perhaps it is the case that the move toward making calculus the standard freshman mathematics course happened during just the period that the problems in the teaching of mathematics in high school were beginning.

3.5.2 Class Size

Seventeen schools reported that class size in calculus was not more than 40 and 7 said that it was 80 or larger. Several schools mentioned that they had gone to large lectures in the 70's and had returned now to small sections. One school mentioned that a comparison had been made and that there was no difference in student performance with classes of 80 students than with small classes and another school reported that students taught in classes of size 100-150 "did as well or better" than students taught in classes of 30-35.

3.5.3 Remedial Mathematics

Also unsurprisingly, many respondents from all but the highest quality universities mentioned the remedial mathematics load in their departments. The factors mentioned just above - calculus becoming a freshman course and problems in high school mathematics teaching - have conspired together to create a heavy load in pre-college mathematics teaching for many departments. Another factor contributing to this is the broader spectrum of students required to take mathematics in college by their major departments. How important this factor is compared to others in necessitating remediation and in changing the content of calculus courses is impossible to judge.

3.5.4 Other Courses

As noted earlier, we received rather little information on any mathematics course except calculus. What information we did receive indicates that courses such as linear algebra, differential equations and abstract algebra have changed even less than calculus. Yes, linear algebra courses sometimes use modern textbooks which emphasize numerical and algorithmic linear algebra more than older textbooks and, yes, differential equations courses sometimes actually cover the numerical methods chapter in the book. But, overall the changes seem to have been few.

The most noteworthy change in undergraduate mathematics courses has been the introduction almost everywhere of a discrete mathematics course or two. But except for computer science majors, these are seldom required courses.

3.5.5 Textbooks

What information we received about texts indicated that those which are well known to be or to have been popular (e.g. Thomas, Swokowski) have indeed been widely used. Although there has been some vocal complaint about the size of current calculus texts and the number of routine exercises in them, we discerned no complaints about this in our sample. It may well be true that as the population of undergraduates studying calculus has broadened, many faculty have welcomed - or, at least, have not disliked - books which present things in bite-size chunks and which provide plenty of the kinds of problems which almost anyone can solve.

4 Interpretations and Conclusions

4.1 The myth of the "good old days"

It is true that many mathematicians have fond memories of undergraduate mathematics courses in which students appeared to be learning a great deal. But how reliable is this vision into the past? How accurate are the undergraduate memories of professional mathematicians as compared, for example, to reports we all hear at cocktail parties from those for whom "math was always my worst subject"?

Indeed, it is not even the case that all mathematicians remember the good old days in the same way. One of the authors of this report remembers the difficulty he had, as a freshman, with limits and continuity. He recalls asking his teacher on many occasions for an example of a discontinuous function. Every time the response was the same, a step function. This reporter remembers thinking that the step function was certainly different from every other example he had ever seen — it was "straight" as opposed to "curvy" and therefore, continuity had something to do with curvature. A few years later, when these concepts were becoming clear, he realized that if only the teacher had once given an example of a "curvy" function with a jump, then he might have started understanding a little earlier.

But our point is that we do not have to rely on anecdotal memory alone. We suggest that the data generated in this study, however sparse, incomplete, and statistically unreliable, does tell us something.

The Final Exams discussed in Section 3.2.1 seem to point in a specific direction. Throughout the 45 year period from 1945 to 1990, the overwhelming majority of exam questions can be categorized as symbol manipulation and calculation, almost none of which required significant thought. In Calculus I, such questions ranged from 65-75% of the total. In the following two semesters of Calculus it is the same — 75-95% and 75-90% respectively. The remaining questions were fairly evenly divided across time and in the other categories.

This lack of variation is repeated in our analysis, in Section 3.2.2, of course content as indicated by published syllabi. According to these reports, discussion of limits has increased over time, consideration of applications has remained fairly constant, analytic geometry has declined and the techniques of integration have been a constant part of Calculus for 30 years. In general, very little change was reported.

Finally, the comments on level of sophistication that we asked for and received (see Section 3.2.3) are in the same vein. It is true that statements that the course has not changed again represents only the memory of mathematicians who have been around for a long time, but the fact that these individuals so rarely remembered any specific changes may perhaps be considered as a corroborating fact.

We feel that, taken together, all of this data is consistent with the contention that our beginning undergraduate courses, anyhow calculus, never did, and still do not function at a very high level of sophistication. In this respect the courses have not changed very much over the last few decades. They are, however, possibly getting worse because the student body they are serving is younger, may be less well prepared, and needs of a course more related to the application of mathematics to other subjects.

4.2 Mathematics in a changing world

Suppose it is the case that, in fact, not very much has changed in the teaching of calculus in the past three or four decades. In particular, suppose that students of similar abilities and intending to major in the same subjects are not taught very differently now than they were in 1950 or 1960 or 1970. Certainly our data does not prove that this is the case. But our data give us no significant reason to believe the contrary. What would be the implications of this conclusion for lower division undergraduate mathematics education?

If not much has changed over three or four decades it must be for one or more reasons like the following:

- mathematicians are generally satisfied with the subject matter of calculus and other lower division mathematics courses and the pedagogy used in these courses.
- mathematicians are not satisfied with the subject matter and/or the pedagogy but in the absence of significant incentives - or pressures - to change either, they have just continued to do what comes naturally and easily.
- mathematicians (or anyone else, for that matter) don't know enough about pedagogical methods that will improve learning.

We submit that it is troubling if any of the above are the case for a major portion of American and Canadian mathematics faculty. This is not the place to discuss the publish-or-perish pressures which may lead faculty to spend as little time or effort as they can on teaching and the preparation for teaching. Nor is it appropriate here to consider the value of research in teaching and learning or the development of innovative teaching methods. How much such issues account for the lack of change in the teaching of calculus, we leave to the imagination of our readers.

But we believe it is even more troubling if the lack of change is a result of a belief that there is no compelling reason to change. The past 40 years have seen major changes in how and why mathematics is used in science and technology. This is not just because of a continuing acceleration in the production of new scientific and technological knowledge. With respect to mathematics, particularly, computers have wrought change in:

- how mathematics is done by scientists and engineers;
- what kinds of problems are solved (i.e. a gradual increase in the relative importance of modern (discrete) applied mathematics and a gradual decrease in relative importance of classical analysis)

The practicing engineer or scientist typically uses mathematics today quite differently than 40, 30 or even 20 years ago. Almost no numerical computation is done by hand. The amount of symbol manipulation done by hand has also decreased considerably and this trend will accelerate in the next decade or so. Does this mean that the contents of undergraduate mathematics needs to change to meet the needs of a new generation of scientists and engineers? Perhaps not, but the burden of proof should not be entirely on those who wish to promote change but rather should be shared by those who wish to defend the status quo.

Of course, the change in the practice of mathematics by mathematicians themselves has changed much less than the practice by those for whom mathematics is a tool. Mathematicians have been much slower to use computers than other scientists and engineers. Although almost all applied mathematicians now use computers routinely, very few pure mathematicians use them. More important, perhaps, a quite small fraction of university teachers of mathematics use computers for anything more than word processing and email.

Here perhaps is the nub of the problem: Mathematics courses and mathematics teaching still seems to serve mainly the needs of mathematics faculty and may or may not serve well students in other disciplines who are required to take mathematics courses. Thus, for example, discrete mathematics is still widely taught in computer science departments because computer scientists have either been unwilling to expend the effort in, or have been unsuccessful in, persuading their colleagues in mathematics departments to teach the kind of courses they want.

4.3 Some characteristics of undergraduate mathematics education

In addition to the level of sophistication, there are a number of other features of our beginning undergraduate courses to which our data relates and we summarize here what was reported.

Traditional lecturing remains the predominant activity in these courses. Class size is, in general, much larger today than it was 30 and 40 years ago, but the feeling on what effect this has had is mixed.

Although there are many indications independent of this study that the use of technology in teaching undergraduate mathematics courses, such as calculus, is rapidly increasing, this is not reflected in the reports we received. One possible conclusion is that this growth has indeed been extremely rapid, but only in the last year or two.

There seems to be general agreement (with a small amount of dissent) that student preparation for calculus has declined strongly over the last few decades. This might be connected with a continued low level of sophistication in that one does not need much preparation for a course that is little more than an exercise in symbol manipulation.

We were not surprised to read many responses that reported a heavy and increasing load of remedial courses, but we did not receive a large number of complaints about Calculus text books. Whether faculty like the existing set of texts or not, it may be that some sort of stasis has been achieved.

4.4 Summary

If we were going to find evidence of striking changes in the teaching of calculus, we would have expected to find it at least in course syllabi and in the questions asked on examinations (Sections 3.2.1 and 3.2.). In the case of final examinations, at least, we had a considerable sample of questions so that lack of data was not a problem. Our results give no support to those who believe that there has been a trend over some sustained period(s) during the last 40 years to make calculus a less-demanding more mechanical, symbol manipulation only course since there is just no evidence of this at all from our data. If anything, the data we have collected points to the conclusion that little such change has taken place.

True, examinations do not always faithfully reflect what was taught. Thus, even when aspects related to higher level thinking may play a major role in the course - and homework - they may appear seldom on examinations since such matters are hard to test. On the other hand, it has been argued (National Research Council, 1990) that, in the long run, if there is a consistent pattern in the tests given in a course, then that pattern will come to reflect, if not what the faculty is trying to teach, surely what the students will make an effort to learn.

It is the case that there are other factors which may be at work here, and these are discussed in the rest of Section 3. But since we have found no evidence at all of significant change in the teaching of calculus over the past few decades, it may be suggested therefore, that the curriculum reform movement needs to look more towards innovations for the future than to search for methods of recreating the past.

5 Acknowledgements

Projects like this require the help of many, many people. Thus we would like to express our thanks to:

1. Mark Johnson and John Norris who organized and massaged much of the data we received.
2. Nicolas Goodman, William Ted Martin, and Gail S. Young who commented on an early version of Appendix 1 and prevented us from sending a much too detailed questionnaire to participating departments.
3. The chairs and other participating faculty at the 53 departments who responded to our request for data as well as the 8 mathematicians who responded to the questionnaire in Appendix 3.
4. In particular, Gail Young, who in addition to helping us with the original questionnaire gave us advice on a number of occasions from his vast store of wisdom on matters related to this task and who spent half a day with one of us answering questions and helping us with this report.

Appendices

Appendix 1

September 1989 letter.

Dear ,

Thank you for your positive response to our letter in June about our Exxon Education Foundation project to do a retrospective study of (mainly lower division) undergraduate education in the United States and Canada. You have expressed a willingness to cooperate with us in gathering the data for our study. This letter will tell you just what it is we would like you to do.

Well, it won't, in fact, tell you "just" what we want. This is an open-ended project to which you may contribute as little or as much as the time and interest of you and your colleagues allow. We could have designed a questionnaire to guide your response but you see too many such questionnaires already. Anyhow we have decided that any questionnaire would be too constraining for the kinds of responses we hope to elicit. Therefore, our request is for a "free form" response as described below. If this leads to information overload, so be it. We will have a graduate student to preprocess the responses for us.

On the attachment to this letter are two pages describing the kind of data we are interested in. Several comments:

1) We are interested in any relevant facts about undergraduate mathematics education over the past, say, four decades. Of course, we realize that you are unlikely to have much in the way of syllabuses or exams or textbook lists or student numbers etc. before, say, 1970 (but please send them if you have them). On the other hand, some of the more senior (yes, older) members of your department may have files or just anecdotes which would be valuable to us. We'd rather have too much than too little. Naturally, the more you can organize the material you send us, (in, for example, tables), the more it will simplify our task. But, organized or not, send whatever relevant material you can put your hands on.

2) Please involve as many of your colleagues as you can and also whoever at your college or university has historical data on course, major and college enrollments.

3) Obviously it won't be useful to receive data, even if you have it available, on each year in the past 10 or 20 or so. Most of the things we're interested in change little from year to year but occasionally change abruptly (textbooks, or use of technology, for example). It would be most useful to us if you choose characteristic years in, say

a decade or half-decade or years in which something distinctive changed.

4) Because of the unspecific nature of our request, we suspect many of you will have questions about how congruent your thinking on this project is with ours. For example, you may wonder why we didn't mention some aspect or other of undergraduate mathematics on the attachment. Like as not, it was our oversight rather than planned. Please, please call one of us to discuss any of your concerns or to clarify anything in this letter or on the attachment (Dubinsky: 317-743-4035; Ralston: 716-636-3186). Both of us have answering machines at these numbers and we'll be happy to return your calls.

5.) As we noted in our previous letter, we expect to visit a few departments after we assimilate the responses to this letter in order to elicit further information when the response suggests that this might be useful. If you think a visit to your department by one of us would be valuable to us (and, maybe, to you, too), don't hesitate to suggest such a visit when you respond.

If your responses are going to be of maximum use to us, we need to have them by 1 December 1989. We hope to massage the data between then and, say, 15 March 1990 and then make the visit referred to above in April or May.

By the time you've read this letter and the attachment, you'll realize that giving us everything we'd like could really be a major project for you. So, please, as mentioned above, do just as much as you and your colleagues can manage. We think that the totality of what we get sent should cover all the areas we're interested in. We'll appreciate anything you send us.

We're looking forward to working with you. With your cooperation — and with just a little bit of luck — we may be able to contribute usefully to undergraduate mathematics reform by supplying the perspective from the past which should inform changes to the present.

Please send your response to: Ed Dubinsky at the Department of Mathematics, Purdue University, Mathematical Sciences Bldg., West Lafayette, IN 47907.

Sincerely,

Ed Dubinsky	Anthony Ralston
Professor of Mathematics	Professor of Computer Science
and Education	and Education
Purdue University	SUNY at Buffalo

P. S. Perhaps for some reason or other you now feel unable to work with us. If so, please let us know right away because we shall wish to invite another institution similar to yours to participate.

UG MATH NOW AND THEN INFORMATION FOR THE STUDY

1. First of all, we would like to know about your institution and how it has changed over the past few decades. We are interested in:
   - Undergraduate Enrollment
   - Number of Mathematics Faculty
   - Number of Mathematics Majors
   - Number of Majors in other subjects such as:
     - Engineering
     - Physical Sciences
     - Management
     - Computer Science
     - Education
   - Quality of Undergraduates (SAT, ACT Scores)
2. The specific courses this study is concerned with are: Calculus, Linear Algebra, Differential Equations, Abstract Algebra and Discrete Mathematics. In the case of Calculus, we need to know which kinds of Calculus course (e.g. audience) you are reporting on.
3. For each of the courses listed under 2., we would like certain information about its intended and actual content throughout the period in questions. This could include:
   - Copies of Syllabi
   - Catalogue Descriptions
   - Final Exams
   - Names of Textbooks
4. We would also like some information about the level of sophistication at which each course was taught over this period and how it has or has not changed. Specifics in this direction include:
   - Rigor
   - Proofs
   - Applications
   - Techniques

   For this item, anecdotal evidence may be particularly important.

5. For each course, give us some information about the students who took it throughout the past few decades.
   - Class Sizes
   - Majors
   - Other information such as level of students (Fr, So ...)
6. Say something about the various teaching methodologies which have been used at various times during this period. You might mention
   - Lecture
   - Recitation
   - Moore Method
   - Socratic
   - Small Groups
7. How about results? Have grade patterns or attrition rates changed or remained relatively constant? If the former, how would you describe the changes?
8. Finally, we would like to know about the use of technology in these courses and how it has or has not grown (or should) during this period. We are thinking of tools such as:
   - Chalkboard
   - Overhead Projectors
   - Calculators
   - Computers
   - Videos
   - Closed Circuit Television

   How have these been used in class, in labs and for homework assignments?

One final comment. In all of the above, we are hoping for a mix of numerical, qualitative and anecdotal information. We don't expect all of you to provide thorough information in each category for each course and throughout the entire period. We do hope, however, that you will each provide enough data so that the union of the information will present a reasonable picture of the first two years of undergraduate mathematics - now and then.

Schools that responded to our questionnaire.

Alaska, University of, Anchorage

Bowdoin College
Brockport, SUNY at
Carnegie Mellon University
Chicago, University of
Colorado School of Mines
Davidson College
Dowling College

Florida A & M
Georgia, University of
Grand Valley State University
Houston, University of, Downtown
Johns Hopkins University
Loyola University Chicago

New Mexico State University
North Carolina, University of, Greensboro
Ohio State University
Princeton University
Rhode Island, University of
San Francisco State University
Saskatchewan, University of

Southern University and A & M College
Towson State College
Trinity University
Washington and Jefferson College
Western Ontario, University of
Yale University

Arkansas, University of, Fayetteville
Brock University
Carleton College
Centre College
Colgate University
Connecticut, University of
Denver, University of
Fairleigh Dickinson University, Teaneck
Gallaudet College
Georgia Institute of Technology
Hood College
Illinois State University
Knox College
Maryland, University of, College Park
New Mexico Tech
Northern Arizona University
Pepperdine University
Queen's University
Saint Peter's College
Sarah Lawrence College
Southern California, University of
Toronto, University of
Trinity College
Victoria, University of
Washington State University
Wright State University

Appendix 3: QUESTIONNAIRE

QUESTIONS ON THE TEACHING OF CALCULUS

1. Do you really think calculus is generally taught worse today than it was, say, 10, 20, or 30 years ago?

   If yes, in what way is it worse taught?

   If yes, do you think the situation is equally bad at research universities, quality four-year colleges and weaker public and private colleges?

2. Please comment on each of the following possible causes of a decline in the quality of calculus instruction:

   A broader spectrum of students taking college calculus.

   Generally poorer preparation for calculus in high school mathematics.

   The refusal by most mathematicians to use new techniques in their teaching, particularly computers in the classroom and outside it.

The refusal by most mathematicians to adjust the content of calculus to reflect developments in science and technology, particularly computers and computer science.

The increasing tendency, at research universities particularly, for faculty to pay less attention to teaching than heretofore.

Anything else?

3. Please comment on any other aspect of teaching calculus or other college mathematics that you think would be useful to our study.

Appendix 4: EXAMPLES OF EXAM QUESTION CATEGORIES

A1

Calculate

$$\lim_{x \to -3+} \frac{x^2 - 9}{x + 3}.$$

A2

Find dy/dx at $x = 1$ if

$$y = \sqrt{\frac{x+1}{x^2+1}}.$$

A3

Find dy/dx when

$$x^3 y + xy^3 + \cos(x + y^3) = 7.$$

A4

Calculate

$$\int \frac{(x^3 + 6x)dx}{x^2 + 4}.$$

B1

State the Direct Comparison Test for the convergence or divergence of a series.

B2

Use the precise (formal) definition of a limit to show that

$$\lim_{x \to 3} 2x^2 = 18.$$

C

A page of a book is to contain 27 square inches of print. If the margins at the top, bottom and side are 2 inches and the margin on the other side is one inch, what size page would use the least paper?

D1

The law of the mean asserts that there exists at least one point in a given interval that satisfies a certain equation. Write this equation. Then for the function $f(x) = x^2$ and the interval $1 \le x \le 3$, find such a point.

D2

State and prove the extended mean value theorem for functions of one variable.

or

Prove that, if $0 \le a_n, 0 < u_n$ and $a_n/u_n \le M$ for $n = 1, 2, 3, \ldots$, then $\sum a_n$ converges if $\sum u_n$ converges.

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