On Student Understanding of AE and EA Quantification

Ed Dubinsky
Georgia State University
Atlanta, Georgia

Olga Yiparaki
Agnes Scott College
Atlanta, Georgia

Abstract

We discuss student understanding of AE ("For all ... there exists ...") and EA statements ("There exists ... for all ...") in natural language and in mathematics. Students were given a questionnaire with eleven declarative statements and were asked to decide whether the statements were true or false and explain their answers. We discuss conclusions based on their written responses and on subsequent interviews with some of the students. We found that students are not inclined to use the syntax of a statement in order to interpret it, particularly if they do not understand it very well; rather, they use the context of a statement to discuss their own opinions about the topic in general, not the actual statement. We also found that students are more inclined to interpret English statements as AE rather than EA. Most students in this study could not distinguish between AE and EA statements in mathematics and did not seem to be aware of the standard mathematical conventions for parsing statements. Finally, we discuss the use of quantifier games as a pedagogical tool to help students understand statements with multiple quantifiers.

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1 Introduction

We do not see things as they are,
we see things as we are.
—*Anais Nin*

Does language govern our thoughts? Or do we form language so that we can express our thoughts? Can a concept exist if we do not have a name for it? Or is the creation of a name required to create a concept?

The world would be a very simple place if papers such as this could answer such big questions. Even though these questions necessarily lurk in the background, this study has a much narrower scope. We are interested in the difficulties students have understanding statements in mathematics that involve existential and universal quantifiers. Our starting point was the idea that students can make sense out of statements in natural language about everyday situations even when there is quantification. Our hope was that if we could learn something about how people make sense of quantified statements in normal discourse, perhaps we could use what may be natural modes of thinking to help them understand similarly structured statements in mathematics.

Our results suggest that the depth of student understanding of quantification in normal discourse is not very strong and may not be a powerful resource for helping students understand quantified statements in mathematical contexts. Moreover, we present results that suggest that what understandings students do appear to have of quantified statements in everyday situations may not transfer very easily to mathematical situations. We will argue that the language we use in mathematics obeys certain rigid rules that our students do not necessarily pick up on their own, and hence we need to help our students learn how the language of mathematics works in order to communicate with them. Thus we will argue that as teachers, instead of trying to make everyday life analogies between ordinary English statements and mathematical statements, perhaps we should remain in the mathematical contexts and concentrate our efforts directly on helping students understand mathematical statements in their natural mathematical habitats. We will further argue that one ends up making analogies even inadvertently, simply by using the natural language to communicate a mathematics statement, and that it is for this reason that we need to teach our students that the language of mathematics works differently from natural language.

Throughout this paper, when we refer to *syntax* we will mean *the arrangement of and relationships among words, phrases, and clauses forming sentences*; when we refer to *semantics*, we will mean *the interpretation or meaning of the words in a sentence and the sentence as a whole.*

We are interested in the extent to which students are inclined to use the syntax of language in order to understand mathematics. Perhaps from the point of view of a mathematical logician there is little distinction between mathematics and formal language, but from a mathematics education perspective it is important to draw a distinction between semantics and syntax.

It would probably be accurate to say that most of our mathematics students (at least in a typical U.S. university or college) do not take a course in mathematical logic, or one that directly addresses the syntax of mathematical discourse, or indeed any course that tries to focus on making sense out of complex mathematical statements. One possible exception to this are the so-called "bridge courses" which have been receiving increasing attention in recent years. Although these courses vary, many have the explicit goal of helping students move from studying mathematics only by learning how to perform various computational algorithms to constructing proofs. It may be that in some cases this is done by paying attention to the syntax of mathematical statements as a tool for understanding the point the statement is making. In any case it is this attention to syntax that is the main concern of the present study.

Most mathematicians would probably agree that we cannot hope to teach our undergraduate students any mathematics (which includes understanding and doing proofs), if they do not know how to read and interpret the language of mathematics. Yet there seems to be a substantial gap in communicating mathematics to our students. While struggling to follow a proof or to come up with one, our students often fail and get frustrated, or worse: they do not even know whether or not they have succeeded.

Motivated by our desire to understand some of the reasons responsible for this gap in communication, we designed this study to address the needs of our mathematics students and any mathematics educator who experiences the frustrations described above. By using students with diverse mathematical backgrounds, we intend to address some of the universal difficulties they experience.

What comes first when people read a mathematical statement: an understanding of syntax or an understanding of semantics? Perhaps in the case of a mathematician reading a mathematical statement this is like asking the chicken-and-the-egg question, as these are intricately linked. Nevertheless, we would argue from our own mathematical experiences that one powerful tool a mathematician uses in understanding a complex statement is an analysis of the semantics based on the syntax of the language in which the statement is given. Is this a "natural" thing to do? In this report, we present data indicating that in the case of a student trying to learn mathematics, the most natural direction is to go from semantics to syntax and that students often are not able to use the syntax as a tool for understanding. Our data suggest that what students do is respond to a statement more or less globally and imagine a context or "world" which the statement is taken to describe. Our results are consistent with cognitive science research into logical thinking (see for example, Johnson-Laird, 1983, 1995 and the web page http://www.cogsci.princeton.edu/ phil/) where it is argued that students think about a syllogistic reasoning tasks in terms of a mental models which is analogous to our notion of world. This world must be very close to the concrete experiences of the student who uses it in place of the syntax to make sense of the statement.

Not only is it possible to go from syntax to semantics, it is also formalizable. The theory of Denotational Semantics, inspired by Strachey's work (see Strachey, 1966) and developed by Scott (see Scott and Strachey, 1971), provides a framework for constructing mathematical models for programming languages. Using semantic equations, one translates a formal sentence; this translation is driven solely by syntax and it produces semantic objects (numbers, funtions, tuples,etc.) for all sentences. Though, of course, most mathematicians probably do not even come close to using such a formal approach, they routinely construct 'semantic objects' or mathematical worlds when attempting to support or refute a given statement. In other words, one of the important skills a mathematician employs regularly is the ability to go from syntax to semantics with an open and creative mind, without necessarily prejudging how reasonable or unreasonable a statement appears to be based on mathematical worlds that are already known, established, and familiar.

The problem with novices then is that their restriction to familiar worlds is limiting and is not sufficient to understand very much mathematics once it goes beyond the most elementary. Imagine, for example, a student in a first course in analysis (or a bridge course) trying to understand the theorem that a continuous function on a closed finite interval is uniformly continuous. It is first necessary for the student to understand the distinction between continuity and uniform continuity. Epsilons, deltas and choosing them have traditionally been difficult for students. In part, this is perhaps because many students are uncomfortable with algebraic manipulations, but our data show that students have difficulties understanding the statements themselves (such as the mathematical statements defining continuity and uniform continuity). How do we help our students? It is not clear that examples and analogies will always help. It may be that the student must learn to use the structures of formal statements to make sense out of mathematical situations in contexts that are new and unfamiliar.

We will argue in this paper that instead of focusing pedagogy entirely on the situations already familiar to students, we should also help them learn to use syntax to to understand complex mathematical statements. Our study and others (e.g., Barwise and Etchemendy, 1991, Dubinsky, 1997) suggest that it is possible to design instruction using a number of different approaches that will help students develop in this way.

This study was designed to deal with a very special case of syntax difficulties: quantifiers. We were motivated by two basic facts. First, it seems that practically all interesting mathematical statements have at least one universal and at least one existential quantifier — and the order in which they appear is crucial. Second, students have great difficulty understanding such statements. Regarding the latter, although it seems to be a part of "conventional wisdom", there is not much in the way of research data to support it. One of the results of this paper is to provide such data.

1.1 AE and EA statements

A formal statement of the type $(\forall x)\ (\exists y)\ R(x, y)$, where R is a binary and quantifier-free predicate, is called an AE statement. Similarly, a formal statement of the type $(\exists c)\ (\forall d)\ S(c, d)$, where S is a binary and quantifier-free predicate, is called an EA statement.

Throughout this paper we make references to AE and EA statements. Most of the statements we refer to are not formal, however. When we say a natural language statement is AE (EA), we mean that if the statement were to be formalized, then it would be an AE statement (EA statement). For example, we refer to the statement *every pot has a cover* as AE because we would formalize it as

$$(\forall p)\ (\exists c)\ R(p, c)$$

where p ranges over all *pots*, c ranges over all *covers* and $R(p, c)$ means c *covers* p.

Of course, natural language is not as formal as mathematical language and the former can be ambiguous as to whether the meaning is an AE or an EA. For example, we refer to the statement *Someone is kind and considerate to everyone* as EA because we would formalize it as

$$(\exists p)\ (\forall q)\ K(p, q)$$

where p, q range over the set of all people and $K(p, q)$ means p *is kind and considerate to* q. Others, however might interpret the statement as an AE and there is some ambiguity. Indeed, one of the reasons for using formal mathematical language is to eliminate ambiguity. One of our interests in this study is to see how students deal with such ambiguity in natural language.

1.2 Description of the paper

We begin this report with a description of the study that we conducted, the students involved and the instruments used. Then we present our results concerning the knowledge and understandings students appeared to have. Although we tried to probe fairly deeply into the thinking processes that students used, we did not do very much at this point to try to improve their understanding. We continue with a discussion of one device we did use to try to improve student understanding — asking them to play a mental game related to a statement. We present the results of this activity as well. Finally, we summarize our conclusions and make some pedagogical suggestions.

2 Description of Study

Our study has two components: a questionnaire and follow-up interviews. We describe these two instruments and the students to whom they were administered.

2.1 The students.

A total of 63 students responded to our questionnaire. Their backgrounds varied. There were freshmen, sophomores, juniors, and seniors. All were in some mathematics class: some were taking a first-semester linear algebra course, some were taking multivariable calculus, while some were taking a junior-level class as preparation to teach middle school mathematics. Almost all students were mathematics or mathematics education majors; the rest were science or engineering students. About half were students in one of two large public universities and the other half were attending a small liberal arts college. Finally, nine of the students were in an advanced senior level/beginning graduate level abstract algebra class; we will refer to this small group as "the graduate students". We will refer to the remainder as "the undergraduate students".

2.2 Questionnaire.

We devised eleven declarative statements and asked students to decide whether each statement was true or false. Students were told that there are no correct or incorrect answers; all they had to do is decide for themselves the truth or falsity of each statement. They were given some space below each statement to write a few sentences explaining their reasoning.

2.2.1 The statements.

The actual questionnaire is in Appendix A at the end of this paper, but for the convenience of the reader we include the eleven statements in compact form. In parentheses, before each statement, we indicate the label (AE or EA). These labels were not part of the original questionnaire given to the students.

1. (AE) Everyone hates somebody.
2. (AE) Every pot has a cover.
3. (EA) Someone is kind and considerate to everyone.
4. (EA) There is a mother for all children.
5. (AE) All good things must come to an end.
6. (EA) There is a magic key that unlocks everyone's heart.
7. (AE) All medieval Greek poems described a war legend.

8. (EA) There is a perfect gift for every child.
9. (EA) There is a fertilizer for all plants.
10. (AE) For every positive number a there exists a positive number b such that $b < a$.
11. (EA) There exists a positive number b such that for every positive number a $b < a$.

(The reader might have been inclined to add a comma to the last statement above; we will address this issue below.)

Our intention was to give students several examples of AE and EA statements in natural language and two in mathematics. We view Statements 1, 2, 5, 7, and 10 to be AE statements, and the remainder (Statements 3, 4, 6, 8, 9 and 11) to be EA statements. So, when we make observations about what students did or did not do with AE statements, we are referring to Statements 1, 2, 5, 7, and 10, and the corresponding list for EA statements. (We will address the reader's possible disagreement(s) with these assessments in sections 2.2.2 and 3.1.)

Given that this study intended to investigate students' understanding of quantifiers, we did not want to complicate the statements any further. Thus, we avoided implications, logical conjunctions, and logical disjunctions within each statement, and so on. Furthermore, we tried to choose statements that would be short and sound reasonably natural in ordinary English (Statements 1 through 9).

The natural language statements (1 through 9) were intended to provide some variety of familiar statements (such as Statement 2 and 5 that are almost colloquial proverbs) as well as some variety of familiar notions involving debatable statements (such as Statement 1, for example, where the notion of *hate* is familiar, but opinions on the truth or falsity of the statement would be expected to vary). Moreover, Statement 7 was chosen to involve the familiar notions of *poem*, *war legend*, etc., but our hope was that most students would not be able to rely on any factual knowledge about medieval Greek poetry, given the obscurity of the topic. This statement was intended to be easy to understand and harder to assess.

Finally, the only two mathematical statements, number 10 and 11, were chosen to meet the above criteria of simplicity in logical structure and brevity; they were intended to rely on practically no mathematical knowledge from any particular course other than some understanding of some concept of *positive number* and the notion of *less than*. (Statements 10 and 11 have appeared elsewhere as well (Dubinsky 1991).

2.2.2 Language conventions and the authors' AE and EA labels

Why did we assign the AE and EA labels to the eleven questionnaire statements as mentioned earlier? Perhaps not many people would argue with the label 'AE' to classify Statement 1, or the label 'EA' to classify Statement 3 or Statement 11. We would expect that more people would argue that the label 'EA' is not necessarily appropriate for Statement 4. After all, perhaps in everyday speech, we frequently use the statements *There is a mother for all children* and *All children have a mother* interchangeably, yet in this study we would label the first statement as EA and the second as AE. Why? It is not that we wanted to ignore the speech of everyday language, quite the contrary.

We believe that, based on our society's conventions, most natural language statements are interpreted based on a number of factors, context being a very prominent one; syntax typically takes a secondary role in aiding interpretations. Natural language comes with a large set of conventions and the issue of exactly how one interprets a given statement is rather complex and certainly not the focus of this work. As a mathematician, however, one necessarily becomes aware of at least some of the natural language conventions in interpretations simply because in mathematics the conventions are different and often in opposition to those in English or other natural languages. We all know the difference, for example, between the typical meaning of "or" in English (exclusive) as opposed to its meaning in mathematics (inclusive). In English, when we state "If I get my paycheck tomorrow, I will buy you a present", we most likely mean what would be expressed as an "if and only if" in mathematics. Examples like these abound and we verify the conflicts in conventions frequently when our students are puzzled by our seemingly peculiar use of logical connectives. Yet we typically deal with these convention conflicts with relative ease because they are easy to identify and easy to address: for example, we can easily tell when our students misinterpret "or" and we can quickly teach our students that "or" is inclusive in mathematics, by definition. Mathematical conventions don't stop there, however. There are firm rules that govern mathematical discourse that go beyond individual words. These rules are often in conflict with conventions of the natural language we use. Can we always tell what may be confusing our students – as easily as we can tell in the case of the meaning the word "or" – when the problem comes from parsing conventions our students do not know? We believe that many of these conflicts create a silent problem – silent because we typically fail to address these conflicts explicitly when we talk with our students.

As mathematicians, when we talk to each other about the weather, sports, politics, gardening, etc., we almost certainly use the conventions of natural language. When we start discussing mathematics, however, we almost certainly use the conventions of mathematics. We switch between the two conventions with ease and we do not bother to signal explicitly that we are switching conventions, despite the fact that we are probably using the very same natural language in both cases. As mathematics educators, we almost always maintain the same behavior: we switch back and forth, but we don't tell our students that we are doing this. We expect them to catch on to this fact on their own. Our students in the meantime, are in the process of learning the material at hand. We believe that by failing to tell our students that the language conventions are different, we frequently fail to communicate the very topic of our focus: the material at hand. This is not limited to verbal communication, of course. The vast majority of mathematical statements our students encounter are not written in formal language, they are expressed in the 'natural' language of the textbooks they read. These textbooks use the special conventions of mathematics.

One of these conventions is that the order of appearance (reading from left to right) of the quantifiers in a sentence makes a big difference. This important distinction (as well as many others) can slip by unnoticed merely because it is not usually addressed explicitly. Our interest in this study was to see to what extent our students are aware of this convention and if so, to what extent they are likely to apply it (that is, to use the syntax) in any context, mathematical or not. Thus, for the purposes of this study, we classify the natural language statements (Statements 1–9) following the mathematics convention of left-to-right parsing. How appropriate is it for this study to label and treat the natural language statements according to mathematics conventions? How appropriate is it for all of us, as mathematics teachers, to use natural language following mathematics conventions?

We chose natural language statements that would be at least to some extent ambiguous in the sense that many of them have two 'reasonable' interpretations: AE and EA. We are aware, of course, that under ordinary natural language conventions, one of the two interpretations is more likely. Statement 4 is perhaps the clearest example of this, as it was intended. We anticipated that Statement 4 would be much more likely to interpreted as an AE statement. Thus, we agree with colleagues who have expressed criticism about treating statement 4 as unequivocally EA. In fact, this criticism supports our viewpoint: we are misleading our students to think that by using natural language to communicate mathematics, we intend, in addition, to use the conventions of natural language of everyday speech. It is not surprising to us that, as we shall see later, students in our study did not follow the mathematics conventions to interpret Statements 1–9. But what we believe is interesting is that students did not seem to be aware that they were using conventions — any conventions at all. In our view, this is the root of the problem with interpretations: not that our students used a reasonable set of conventions to make statement interpretations, but that they seemed to have available only one set of conventions, and in addition, that they did not appear to be aware that they were applying conventions to interpret possibly ambiguous statements.

Finally, one last comment on statement interpretations. Throughout this paper we refer to students' tendencies or preferencies to interpret statements one way or another. These are not intended as value judgments on a student's reasoning or interpretive abilities, especially if the interpretation is the opposite of our label (AE or EA). We are not suggesting that our label for a statement is "the correct one". We are merely interpreting and labeling statements according to mathematics conventions and subsequently we are discussing when, where, and how often students appear to follow this set of conventions.

2.2.3 Grading the questionnaires.

Every student was asked to provide an explanation for the decision (true or false) he or she made for each statement. We used the student's explanation to decide whether the student interpreted the statement as AE or EA, and given the student's interpretation, to assess the validity of her or his argument. We then "graded" each response with an ordered triple. The first component of the triple represents our understanding of a student's interpretation of the statement at hand: whether the student appeared to understand the statement as AE or as EA. The second component of the triple assesses whether the student's reasoning is valid, given her or his interpretation. Finally, the third mark keeps track of the decision the student made: True or False.

For example, consider Statement 3: *There is a perfect gift for every child.* We believe that for most people there are only two reasonable interpretations: *For every child in the world, one can find a gift that is perfect for that child* (AE), and *There is some gift that would be perfect for each and every child in this world* (EA). Thus, when evaluating a student's answer (AE vs. EA), we compared each response to these two possibilities only; there were very few cases for which it seemed that the student did not have one of these two possibilities in mind.

Here are some examples of the triples we assigned.

- A student circled True, and responded with: *You can always find a gift to give a child that is more desired than any other. The gift may vary from child to child, but there is always something (not necessarily material).* We believe this student interpreted the statement as AE and we assigned the triple $(AE, \text{valid}, \text{T})$.

- Another student circled False, and responded with *I'm not sure what is meant by perfect gift. If it is a treasure of some kind, tangible or not, or if it is a skill, either way I could not limit life to perfectness for every child. I do, however, think that every child possesses a unique talent/skill, individual to the child, but not necessarily extraordinary.* We believe that this student understood this statement as AE and we assigned the ordered triple $(AE, \text{valid}, \text{F})$.

- Another student circled True, and responded with: *The perfect gift for every child is to love and nurture them. Any other material gift they will almost always grow out of.* We believe this student interpreted the statement as EA and we assigned the triple $(EA, \text{valid}, \text{T})$.

- Another student circled False, and responded with: *Some kids hate everything you give them, even if they asked for it, because they are spoiled.* It is not clear whether this student interpreted the statement as AE or EA because the response could be addressing either interpretation equally well. In fact, the response provides a valid reason for deciding that the statement is false in both cases (both possible interpretations). In this case we assigned the triple (unclear, valid, F).

- Finally, another student circled False, and responded with: *There is more than one perfect gift. Children like lots of different things.* It is unclear whether this student was thinking of the statement as AE or EA, and in either case, the response does not provide a valid reason for deciding the statement is false, in fact it seems to provide a reason that the statement (in either interpretation) is a weaker thruth, based on the student's viewpoint. So, we assigned the triple (unclear, not valid, F). (There were very few of this case.)

When devising our questionnaire, we tried to express Statements 10 and 11 in as simple and natural a way as possible. Our original statement for 11 was *There exists a positive number b such that, for every positive number a, $b < a$.* In the end, however, we became concerned that statement 11 would have additional commas that Statement 10 did not have. We wondered whether the punctuation difference would lead students down the wrong path when trying to compare the two statements. So we decided to delete the commas and have statement 11 exactly as it appears in the appendix.

We were then faced with another difficulty that we did not anticipate. Fifteen students misread (interpreted) Statement 11 as *There exists a positive number b such that, for every positive number a, $a \cdot b < a$.* But of course we were not interested in this interpretation. As it turned out, all students who claimed Statements 10 and 11 were the same (or reworded) were among those who did not interpret Statement 11 with a and b multiplied. Thus, despite this mishap, we have been able to make sense out the data we collected for just this reason, as we were interested only in those students who interpreted the quantifier-free part ($a < b$) as the same in both mathematics statements in order to see how they distinguished (or failed to distinguish) between the two statements. When we looked at how students compared Statements 10 and 11, we considered only the responses of the forty-eight students who interpreted Statement 11 without multiplication, as indicated in table 6 of appendix B.

2.3 The Interviews.

After grading the questionnaires, as described above, we divided the students into very rough equivalence classes (based on patterns in their responses). We came up with five different classes, and selected members of these classes for our interviews. However, because the selections were made to assure simple representation of each class, the set of students we interviewed does not reflect the difficulties the students had in the same proportions as the set of students who responded to our questionnaire. In particular, 57% of the students we interviewed were able to assess the truth value of statement 10 correctly, whereas only 41% of all the students who responded to the questionnaire got statement 10 right.

We interviewed a total of fourteen students. All of the interviewed students were undergraduates. Each interview was conducted by one of the authors with one student at a time, was recorded on audiotape, and was subsequently transcribed. The average length of each interview was 45 minutes.

During the interview a student was given the opportunity to discuss the thoughts and reasons that led to each decision about the truth or falsity of each statement (while looking at her or his written responses to the questionnaire). In addition, each student was given the opportunity to revise any written answer — and some students did revise answers upon rethinking some of the statements.

At the beginning of each interview, we explained that we wanted to focus on the given statements and to discuss each student's reasoning that made her or him decide whether a statement was true or false. We made it clear that explaining their reasoning would necessarily involve personal opinions about the topic at hand, and so we would ask them about their opinions, but that we would not debate or focus on any opinions. We would question them on their reasoning to support or oppose the statements printed on the questionnaire, no other statements.

The interviews focused only on statements 1, 3, 6, 9, 10, and 11. Besides asking students to reiterate or expand upon their reasoning for each of these statements, the main questions we asked during these interviews were:

- For Statement 1, replace the word *somebody* with *at least one person* and with *one but no more than one person*. Replace the word *everyone* with *many*, with *most*, and with *a whole lot of people*. All these replacements were done one at a time, not several at once. In each case we asked whether the truth value of the statement changed and whether the statement itself changed in meaning.

- For Statement 3, make similar replacements as in Statement 1 for the words *someone* and *everyone*. Then we asked students to describe how circumstances or the world might change in order for the very same statement to change its truth value. However, we did not use expressions such as "change the truth value". Here is an example of a typical scenario. A student has decided Statement 3 is false and has provided some reasons for this decision. We would then ask *What would it take for you to decide that this same statement is true? Can you describe the circumstances or the world situation that would be necessary for you to decide that this statement is now true?* In case a student had decided this statement was true, we would replace the word *true* by the word *false* in the above questions.

- For Statement 6, we went over each student's reasoning. We then asked whether the statement means there is a key that is intended to be common to all people, or whether the statement allows for different people to have different keys.

- For Statement 9, after going over the student's basic reasoning, we introduced a game and played variations of this game with the student. The game is described in detail in Section 4.

- For Statements 10 and 11, we first discussed the student's basic reasoning that was written on the questionnaire. We then played games analogous to the one introduced with Statement 9. See Section 4 for further details.

2.4 Methodological issues

The work reported in this paper raises a number of methodological issues. There are methodological concerns related to the labels (AE or EA) we assigned to each of the 11 statements, our assessments of the true/false designation the students made for each statement, our assessments of the validity of the student's explanation on the questionnaire, and our interpretations of student comments in various interview excerpts.

We have already described above the precise method (left-to-right convention) used in assigning labels to the 11 statements. Although the reader may or may not agree with our choices, they are precise and repeatable. In the remainder of the paper we have tried to be careful about not making judgements like 'correct' or 'incorrect' of the students' choices of these labels.

Assigning a label to a statement entails an interpretation. For each of our statements, its truth or falsity is quite clear once an intepretation is determined. Again, we have tried to make clear in our assessments of the students choice of truth value the interpretation on which the assessment is based.

Assessment of the validity of the students' explanation of her or his true/false designation is more problematic. It amounts to reading an explanation and deciding whether it is logically correct or not, based on the assumptions the student is making about what is being explained. These assessments were made by one of the authors and checked by the other. We have tried to present examples illustrating our analyses in making these assessments. Of course the full set of questionnaires is available from the authors for any reader who would like to make an independent determination.

Finally, there are the interpretations of interview excerpts. These are entirely the judgements of the authors. We have tried to present enough of each excerpt and an explanation of our interpretation in each case so that the reader can make an indpendent judgement. Again, the full transcripts are available on request.

The main thing that is lacking in our methodology for analyzing the transcripts, as well as, to some extent, the assessment of the validity of the students' explanations, is a theory. It would have been better to begin with a theoretical description of how people think about statements with two different quantifications and then analyze the student responses in terms of this theory. In fact, the present research is preliminary to developing such a theory. It is a first attempt at exploring data on student thinking about these statements. The next step will be to develop a theory and then conduct a similar study that involves this theory both in the design of the study and the analysis of the data. Our hope for the present work is that it will inform such future research.

3 Results

In this section we discuss the performance of the students on the questionnaire and in the interviews. We consider this performance in terms of several issues that seem to be salient in our results: tendencies of students, in dealing the EA statements, to interpret the statements as AE; the contrast between student performance on statements in natural language about everyday situations and statements about mathematical situations; how students deal with a situation when they do not have enough information to decide whether it is true or false; tendencies of students to ignore quantifiers; abilities of students to negate statements or imagine situations in which the truth value of a statement changes; and some examples in which students were comfortable enough with a statement to be creative about it in the form of making a joke.

Although there were only 9 graduate students in this study, there are some comparisons we can make with the undergraduate students. The graduate students had more brief and more succinct answers in their questionnaires (regardless of the validity of the reasons they presented). Furthermore, graduate students were somewhat more likely to give valid reasons for their answers.

Nevertheless, the success rate of even the graduate students was not ideal: two (out of nine) graduate students did not get Statement 10 right, only five of the nine got Statement 11 right. Indeed the overall patterns of responses for these students is sufficiently similar to those of the undergraduates that we do not distinguish between the two groups in the results that we present here. The breakdown of our numbers according to the two groups is available.

Note. We remind the reader that there were 63 students involved in the study, so all sets of students mentioned in Appendix B have size less than 100. However, almost all counts in Appendix B are given in percentages in order to make comparisons easier.

3.1 Interpreting EA statements as AE.

On both the questionnaire and the interviews we found a strong tendency for students to favor an AE interpretation over an EA interpretation. As we indicated in section 2.2.2 we are aware of the fact that the statements we chose are to some degree ambiguous (though some more than others).

More important for the issue of ambiguity is the issue of language conventions that one might use to resolve any ambiguities, and the awareness (or lack thereof) that one is interpreting a statement. In ordinary communication, as in statements similar to 1-9 of our questionnaire, this might not be so crucial. In some other uses of natural language, it becomes fairly important, as any attorney would surely attest. As mathematicians, we know that every detail counts in mathematical statements.

Throughout this study, we welcomed any suggestion or even hint of a suggestion from a student that any of the given statements in our questionnaire was ambiguous. There were no such written responses on any of the questions indicating there was ambiguity. During the interviews, even when we explicitly probed to see whether a student who had interpreted Statements 6 and 9 as AE could see either of them as an EA, many did not. Almost all of the interviewees continued to see these statements as AE and saw no ambiguity. (There was only a single exception to this, where one student, during the interview, said that Statement 9 could be taken to be EA or AE, but then she decided that it was an AE statement.) To pursue this question further, we managed to get some students to agree that Statement 6 or 9 could be interpreted as an EA and then we asked them to suggest wording that could ensure this. In some cases, they gave us the same statement as the one in the questionnaire.

Though, as we noted above, we told students that the focus of the interviews was to discuss the questionnaire statements. Thus, given this focus was cast on the statements, we found it interesting that the students did not examine the statements more closely, even after our questions about whether some statements could be interpreted differently.

In the following two sections we will provide some details regarding this observed tendency for AE over EA in the questionnaire and the interviews.

3.1.1 AE over EA on the questionnaires.

The results from the questionnaire are summarized in Appendix B.

¿From Appendix B, Table 2, we see that 94% of the students interpreted at least one EA statement as an AE. For the six EA statements, we see from Table 1 that the percentage of students who interpreted them as AE ranged from 11% to 81%. On the other hand, from Table 2 we see that only 5% interpreted at least one AE statement as an EA; the percentage of students who interpreted AE statements as EA ranged from 0% to 3%.

Finally, from Appendix B, Table 3, we see that when the student interpretation was clear, the chances of a valid argument were high. In other words, there is a close relationship between making a clear interpretation of a statement as AE or EA and giving a valid argument for the truth value assigned to it. Of course, this is, at least in part, due to the fact that we based our assessment of interpretation on the arguments students wrote to support their decisions (true or false) for each statement. On the other hand, if a student's interpretation of a statement's form was not clear from the arguments they presented, we gave them the benefit of the doubt: if the student's argument could support both the AE and the EA form of the statement (or their negations), we deemed the argument 'valid' (see description of questionnaire grading, section 2.2.2). Hence, we believe it is fair to say that there is a close relationship between making a clear interpretation giving a valid argument.

3.1.2 AE over EA in the interviews.

During the interviews we had the opportunity to confirm and question students about their interpretations of the statements. Students had the chance to revise any previous answers and to expand upon their thinking. We continued to see a strong tendency to interpret statements as AE, even when the interviewer suggested that the statement at hand may have an EA form. Some students avoided dealing with this issue directly, others simply rejected the possibility, and others were not really able to resolve the (possible) ambiguity. In fact, many times during the interviews, students appeared to be expressing their point of view about the topic at hand, not really focusing on the particular statement. For example, a student would look at Statement 1 and discuss the nature of hate among people without necessarily discussing what Statement 1 *says* about such hate. During these discussions, the tendency toward AE interpretations was strong. Here are some examples.

We begin with an example[1] of a student, AND[2], who is asked for a view of the world that would make Statement 3 true. Her response is actually to change the statement by reversing the original predicate relation. The result is indeed an AE statement which agrees more with the world view that she then describes. The interviewer repeats Statement 3 in its original form, but this does not seem to affect the student. She seems to be talking more about the topic of kindness and less about what the statement says.

> **I:** OK, good. OK let's move on to number 3. Ahm, and here again I want you to tell me how you were thinking. Someone is kind and considerate to everyone. And you said that that's true. So, what view of the world do you have to say that that's true?
>
> **AND:** For everyone is kind to somebody?
>
> **I:** Right.
>
> **AND:** Just ... I mean I know that people ... I am talking from experience, there are people that think they have done something ... or they might not even realize that sometimes, I don't think that people sometimes realize that they have done something, but what they have done could have been so nice to me, that I would have been like 'oh, my gosh!!' ...
>
> **I:** Hm-hm. Now I am interested in the someone and the everyone. What do you say? You are saying that 'someone is kind and considerate to everyone' – you say that's true. What is ... what is it that makes that true?
>
> **AND:** (P) Oh... I guess someone is not kind... well, every person has had some kind of a kind act done towards them. Now everyone hasn't done that act to everyone [student

[1] Throughout the interview excerpts, we use the following conventions. The letter 'I' indicates the interviewer is speaking; (P), (LP), and (VLP) indicate a pause, a long pause, and a very long pause, respectively.

[2] The actual names of students have been replaced by three-letter codes throughout this paper. However, different students are referred to by different codes, and different excerpts from the same interview with any student use the same code.

speaks very deliberately here] ... Because I don't... I can't do a kind act to people that I have never met. I mean like ... or then again, I suppose I could ... I could give money to the poor countries, or you know things like that. But I mean there is no way that everybody can help every other person. I mean I am sure that things I have done have possibly helped people that I don't know, but there is no way it has helped everybody.

I: So, what is it that's happening here? Because you said that's true.

AND: Right. Because I mean someone .... everybody has had something nice. Somebody... everybody has had something considerate done... Whether it'd be a word of encouragement or, ... a pat in the back, or they'd been given food, or shelter. I mean everybody has had something nice done for them or to them.

Next we have another example from the interview with AND who, when asked about Statement 9, seems to respond by focusing on her opinion about the general topic rather than why, in her view, the statement is true. At first, she is not paying any attention to quantifiers at all. Then when the interviewer focuses her on the statement, she seems to be thinking about various situations with various plants and what the fertilizer would be — which is much closer to an AE interpretation than EA.

I: OK. Let's go over to 9. OK... What was your thinking on this one?

AND: Ahm, that God really makes the true... I mean, true there is the ammonia, and nitrates, that fertilize, but without God's nutrients and soil and the rain there wouldn't be anything to grow... and therefore that is the real fertilizer of it.

I: OK. Suppose we consider that all of these things, the nutrients and the soil, came from God, ah... but let's not think about that. Let's think about the nutrients and the soil, and so forth. That's the version of fertilizers . Not where they came from. The statement that there is a fertilizer for all plants, is that true?

AND: True. Because the nutrients and the soil fertilize. I mean they help it grow. Obviously if you just leave the same plant or crop in the same ground, you are gonna deplete the nutrients, so it won't have a fertilizer ... but unless you just do that, it will have its natural fertilizer from the nutrients in the soil.

I: OK. And would that fertilizer in that single plant work for all plants?

AND: No. I mean there are different areas of the world that are more suited to growing different types of ... crops... I mean part of it is the climate ... but part of it too is the soil is different. So it wouldn't have the same amount of nutrients so the crops wouldn't get as much of the nutrients as they needed.

It could be that the previous student had difficulty even understanding an EA version of Statement 9. This is not the case with JUL who clearly understands both possible interpretations and chooses the AE version because the statement says "a fertilizer" rather than "one fertilizer".

I: OK. Great. Take a look at number 9. What about that one? What were you thinking when you did that one?

JUL: (laughs) Ahm.... I don't know, I really didn't know quite how to... what to think about this problem – or this question. A statement, I guess. Ahm, (P)... This one is harder for me. I took it on face value and didn't read into it. I don't know how to really read into it I guess.

I: I am not sure I understand what you mean by face value and how to read into it.

JUL: Well, I mean ... I didn't know if there was a certain like meaning behind it – kind of like in the magic key problem, there is not obviously a certain actual key, so I don't know if you meant an actual fertilizer for all plants or if it was... if that was just kind of (P) I don't know...

I: One possibility then would be that there would be one fertilizer for all plants...

JUL: Hm-hm.

I: What's the other... what's another possibility?

JUL: That's what I didn't know. Yeah.

I: OK. So, is that the interpretation you gave it?

JUL: That there was... that there was one fertilizer for all plants?

I: OK...

JUL: Yeah.

I: And you thought that was true.

JUL: Yeah. (P)

I: Can you tell me why that's true?

JUL: Ahm... That there's ... not necessarily the same fertilizer for all plants, but there is a certain like kind of fertilizer for all plants.

I: But not necessarily the same you say?

JUL: Right.

I: Oh. So that's not ...

JUL: Oh, is that ...

I: ...well...

JUL: OK, no! I guess... I see a difference between: there is "a" fertilizer and there is one fertilizer ... OK, so now ...

I: Right...

JUL: OK, so that's what ... I ... if it said 'one' I would say false. Because it might.. that I would take that as the exact same fertilizer for all plants.

I: And you don't think that's true.

JUL: No.

I: Why not?

JUL: Ahm... plants need fertilizers different fertilizers ...

I: different plants need different fertilizers ?

JUL: Yeah.

I: OK. So what about the other way now, the other interpretation?

JUL: There is "a" fertilizer ... That there is ... there is 'a' fertilizer not necessarily the same one for all the plants, but there is a fertilizer made for all plants...

Finally, we have an example from CAR who, in discussing Statement 6 appears to have an EA interpretation — water being the key for everyone. But she shies away from this, pointing out that it affects everyone differently and although the water may be the same for all, the glass is different. When the interviewer tries to get her to clarify more (with a tendency towards EA by mentioning only the water), the student comes down in favor of the AE interpretation.

I: OK, OK. That makes sense. Now let me ask you something about your interpretation of the statement. You were saying earlier that you saw this as implying that there is something out there, right? That there is this external key.

CAR: Hm-hm.

I: Did you interpret the statement as saying there is this external key that is supposed to be working for each and every person, sort of like a common thing that is supposed to work for everybody, or ... did you interpret it to be ahm ... that for every person there would have to be *some* external key?

CAR: Yeah, I saw it as like an individual thing. Like, am, like in my mind when I read that question I saw something like ... like like, maybe like ... like water, or something, you know, like everyone can have a glass of the water. And that water will like affect them in a different way. You know? Because there is such a mass quantity of it. And every glass is different. That's how I saw it. And when I thought of it that way that like what's inside that cup, I mean like the magic key ... you know... I saw key as like being as like that force. You know? [Laughs nervously.]

I: I think you *are* making sense. What I am trying to understand is whether you thought that the water, or whatever, would be ... ahm ... or *might be* different for every person ...

CAR: Yeah, I think it would be different. It would be the *same* like, in like, a large like scheme of things. Like the same concept, but then it would have different effects on every person. You know?

I: OK, OK, all right. Fair enough. Great. Let's move on to number nine.

3.2 Reasons for AE tendency.

We consider several possible reasons for the observed tendency towards AE interpretations: likelihood of truth of a statement; cognitive difficulties; specific wording of a statement in general; and, the use of the term *all* as opposed to *every*.

3.2.1 Truth of a statement.

One reason for an AE preference might be that more often than not, the statements we make in everyday life are AE and not EA (whenever they involve two different quantifiers, of course).

Another reason might be that people tend to interpret ambiguous statements so as to make them true. Given an AE statement and its EA variation, the EA version, of course, implies the AE version. So the AE version is much more likely to be true, and can be true while the EA version might be false; on the other hand, if the AE version is false then so is the EA version. So a tendency to favor truth would result in a tendency to favor an AE intrepretation.

Contrast the following (see Appendix B, Truth and Falsity): 30% of the AE statements were declared True, but only 10% of the EA statements were declared True. So, it could be that there is a preference for true statements, hence the preference for the AE interpretation which is more likely to make a statement true. Further support for a tendency toward true statements can be given from observing the following discrepancy: while only 11% of the students interpreted Statement 3 as an AE statement, Statement 4 was interpreted as AE by 81%. Statement 4 would be truly absurd (not ‘merely false’) if interpreted as EA, whereas Statement 3 is not so absurd as an EA statement (Appendix B, Table 1).

3.2.2 Cognitive difficulties.

Suppose we are given an AE statement, $\forall x\ \exists y\ R(x, y)$, and a corresponding variation $\exists y\ \forall x\ R(x, y)$. Proving an AE statement is false is equivalent to proving an EA statement is true (since $\neg(\exists y\ \forall x\ R(x, y))$ is equivalent to $\forall y\ \exists x\ \neg R(x, y)$), and vice versa. So, from a mathematical point of view, AE and EA statements have the same degree of complexity.

Despite this similarity between AE and EA statements, a student trying to assess the truth value of an AE statement, versus that of an EA, may encounter more cognitive difficulties in the EA case. Why? Let us contrast the sequence of steps one has to take in order to decide whether an AE statement is true with the corresponding sequence of steps taken for an EA statement. (In each of these two cases, we are not interested in examining the polished and final product of a proper proof here. We are interested in an approximate outline of the mental steps one might go through in order to come up with the ideas that will, hopefully, lead to a proof. Thus we ignore the possibility that we may have infinitely many xs and ys to consider.)

Suppose the AE statement at hand is

$$\forall x\ \exists y\ R(x, y).$$

We might start by considering mentally a single x_0 and asking ourselves whether there is some y_0 (of the appropriate type, depending on the context) such that the relation $R(x_0, y_0)$ holds. Then we iterate this process. As long as the answer to each individual question is ‘yes’, we continue, until we exhaust (mentally) the whole set over which x ranges. If the answer turns out to be ‘no’ at some point, we stop and declare the statement False.

On the other hand, suppose the EA statement at hand is

$$\exists y\ \forall x\ R(x, y).$$

To begin evaluating this statement, we might start with a y_0 and ask ourselves whether all the xs, simultaneously, satisfy $R(x, y_0)$. If the answer is ‘yes’, we stop and declare the statement True. If not, we need to go back and consider another choice for y. Iterate this process until we reach a ‘yes’; if after exhausting all the possible ys we have answered ‘no’, we say the statement is False.

Though both the above sequences of mental steps one might take involve some back-and-forth checking, they differ in one important way. The AE verification process remains at the x-element and y-element level. That is, each comparison is done with one pair (x_0, y_0) at a time. On the other hand, the EA verification process pushes us one level higher: each time we consider a y_0, we need to consider all the xs at once, the whole set of xs that are available. Thus, in set-theoretic terms, the *rank* of the sets has increased and in cognitive terms, it is necessary to see the set of all xs as an object of consideration. Perhaps this explains why one might first give an AE interpretation to a statement involving one existential and one universal quantifier: such an interpretation is easier to understand and easier to assess.

Obviously, as mentioned above, if the AE statement at hand is false, the student ends up verifying that an EA statement is true. But the point is that without knowing the answer in advance, a student can find an easy place to start the verification process. In other words, AE statements, thanks to the order of quantifiers, provide a starting point, a strategy, for a student who wants to begin assessing the truth value of the statement.

EA statements on the other hand do not provide such an obvious strategy. A student who needs to assess the truth value of $\exists y\ \forall x\ R(x, y)$ needs to look for y_0 that works for all the xs simultaneously. If we think of the search for such a y_0 as done totally blindly, this quickly becomes frustrating. One needs to look for a y_0 that is likely to work. But how do we choose such a y_0? It may seem that we are looking for something without knowing in advance what we are looking for. Perhaps this is a difficulty our students experience in dealing with EA statements in mathematics. They are looking for such a y_0 without knowing its characteristics, and without realizing that its relevant properties are given precisely by the statement itself. Thus this pursuit may appear as though one is trying to build a house without its foundation.

There are at least three obvious objections to the above discussion. Consider the EA statement $\exists y\ \forall x\ R(x, y)$. If, in fact, there is a unique y_0 such that $\forall x\ R(x, y_0)$ holds, and this y_0 happens to be the only reasonable choice among the other ys, then a student is more likely to find it without great difficulty. Another way a student may bypass the initial difficulty of not having a beginning strategy for assessing an EA statement is by having enough mathematical maturity to know that existential statements do not require constructive proofs. Finally, yet another way is for the student to think of assessing the negation of the given EA statement instead. These objections, however, do not really address the target population of this study. The first objection involves what is probably a very special case. We believe that the second and third objections involve examples of students with higher mathematical sophistication (and in particular, they are examples of students who must have made substantial progress in mastering AE and EA statements already). Thus, we believe that for the student who is new at dealing with AE and EA statements (and needs to construct proofs involving such statements), the cognitive difficulties described above are real.

3.2.3 Effect of the specific wording.

It is possible that the difficulties the students had in interpreting the statements in the study are dependent on the specific wording which was used. For example, JUL interpreted Statement 6 as an AE on the questionnaire and maintains this view in her explanation during the interview. However, the interviewer is able to reword the statement in such a way that the student sees that it is an EA.

> **I:** OK, great. Have a look at number 6. What do you think about that? Cheesy [fate?] (both laugh)
>
> **JUL:** I don't know, I guess I meant by that ... that there is kind of that in the whole fate of ... there is this person for everybody in the world and some people might not ... I think at one point in everyone's life they ... even if they don't get married, or whatever, there is someone that they ... not necessarily that they fall in love with or anything like that, just that changes their life or makes a big difference in their life... Someone they always remember or treasure... so... I think that's what I meant.
>
> **I:** OK. Suppose we change that statement .... So, for you the magic key would be a person–right?
>
> **JUL:** Yeah. Or not necessarily, maybe a certain event or something, yeah... it doesn't necessarily have to be a person.
>
> **I:** OK. Now suppose we change the statement to read 'there is a magic key common to all [persons?]'
>
> **JUL:** ... common to all... I would take that to mean that it is the same key I guess I would take that to mean that's the same key ...
>
> **I:** The same key...
>
> **JUL:** To ... for everyone's heart, if it's common to all hearts, that it would be the same.... key whatever it was. Let's say something like love was that key.
>
> **I:** I see.
>
> **JUL:** Then love would be the only thing that could unlock everyone's heart I guess. But I wouldn't think that ... that would have to be necessarily true..
>
> **I:** So you think that's false?
>
> **JUL:** Yeah.
>
> **I:** OK . And why?
>
> **JUL:** Because I don't think that there is one particular thing that would ahm... that would unlock everyone's heart because everybody is different. And so, I mean people with different views.... different I don't know, different things are important to people in their lives. And so it's not going to be the same.

Thus we see that this student is capable of interpreting a statement of this kind as an EA but either because of the wording or some other reason, she chose to interpret Statement 6 as an AE.

It is certainly true that the use of extremely specific wording, such as "common to all" or as another student suggested for this same statement, "There is one magic key", could influence students to give EA interpretations of statements. But it may be that this masks the tendency rather than explains it. For example, how does the rewording argument explain the fact that so many more students interpreted EA statements as AE than made the reverse? It is also the case that some students showed in the interviews a strong resistance to interpret a statement as an EA, no matter what rewording the interviewer used. See, for example, the last excerpt in Section 3.1.2.

3.2.4 All and Every.

One specific kind of wording that one might consider as important for distinguishing between AE and EA statements is the choice between using the word *every* (or one of its derivatives) and the word *all*. Perhaps this can make the difference between the AE or EA interpretation of a statement. The idea is that the word *all* seems to be collective, and it makes a statement understood as EA, whereas the word *every* appears to address each case individually, and hence would suggest that the statement is AE. Our evidence, however, does not support this suggestion. This can be seen by comparing Statement 4 (which uses the word *all*) with Statement 8 (which uses the word *every*): 81% of the students interpreted Statement 4 as AE, whereas only about 40% interpreted Statement 8 as AE.

Another possibility is that the use of the word *every* (or one of its derivatives) will signal the reader to interpret a statement as EA. Again, our data does not support this. This can be seen by comparing Statement 3 (which uses the word *everyone*) and Statement 6 (which also uses the word *everyone*): only 11% of the students interpreted Statement 3 as AE, while more than three times as many students (37%) interpreted Statement 6 as AE.

There are many other pairwise comparisons one can make (for example Statement 4 and Statement 9), but it seems that in all such comparisons, the mere choice of words does not make the difference. Perhaps the context is most important. This could explain why 81% interpreted Statement 4 as AE, for example: the EA version of the statement seems absurd. Yet only 38% interpreted Statement 9 as AE, despite the fact that Statement 4 and Statement 9 have identical syntactical structure. In the context of Statement 9 neither the AE interpretation nor the EA interpretation seem as absurd.

It makes sense that our students would favor the context of a statement over its syntax. This is the convention we use in everyday, natural language. We tend to favor interpretations of statements that 'make sense' in the world in which we live. Aside from truth values, we favor the interpretations of statements that do not sound absurd since most people, most of the time, do not make absurd statements. In fact, if we hear or read a statement that strikes us as absurd by some standards of 'normalcy', we often interpret it by replacing it with the closest non-absurd statement we can think of. We all do this, it is part of everyday communications. For example, in the course of a conversation, if someone makes statements that are partially incomplete (or vague), we tend to 'fill in' what is necessary so we can make sense of the conversation. This does not happen all the time, of course, but we typically do this, unconsciously, for otherwise, we would interrupt each other constantly in order to clarify. This principle of implicit interpretation helps us communicate in a normal fashion. We see this principle being used in advertising. For example, commercials can provide us with vague statements that we are inclined to interpret one way (using our convention of 'closest statement likely expressed'), but are in fact intended to mean something else, thereby misleading us. We also see this principle in puns and jokes, where we might interpret statements one way (the conventional one), and ultimately be surprised by the punchline. We can observe this principle operating in many other examples of everyday communication.

A consequence of this principle is that we often end up not being fully aware of the statement at hand, but rather of our interpretation of the statement. This is what we observed in this study. Our students were not so much conscious of the written statement on the questionnaire per se. It was as though the statement was a window from which they were looking out. The students described what they saw looking out the window, but they did not see the window itself. When coming up with an interpretation of a statement, they were not aware of the process they followed in order to reach their interpretation. We will see this later when we discuss the interviews with the students, when they had the opportunity to explain why they interpreted the statements they way they did.

If context is indeed so important in determining one's interpretation of a statement, more so than syntax, then we see why we run into problems with the mathematics statements (10 and 11). We will see that students are on much more shaky ground because they do not have as good an understanding of the world of numbers as, say, pots and covers, and they do not have any everyday-type of conventions to go on in order to interpret these statements. In addition, they typically have not been taught the conventions for interpreting the order of quantifiers in mathematical statements.

3.3 Quantified statements in natural language and in mathematics.

Without a doubt our data suggests that students were much more capable of handling the natural language statements than the mathematics statements. In fact, only 41% of the students got statement 10 right, and only 9% got statement 11 right. (See table 1 and 5 of Appendix B. If we count correct interpretations and justifications of these statements when students considered a and b as ranging over the integers rather than the real numbers, then 49% and 14% of the students got statements 10 and 11 right, respectively. Even so, these are not high rates of success with statements 10 and 11.) Yet, 78% of the students gave a valid argument for seven out of the nine natural language statements (Appendix B, table 4). We will try to look more deeply at this phenomenon by considering some excerpts from the interviews, first in everyday contexts and then in a mathematical context.

3.3.1 Statements in everyday contexts.

Many students approached a strong understanding of simple statements from everyday life. Nevertheless, it is interesting to note that only 25% of the students gave valid arguments for all nine English language statements, and during the interviews, they did not show as strong an understanding on very many statements. We believe that overall, their understanding of the English statements was not disastrous, but neither was it as solid as the questionnaire data would seem to suggest.

During the interviews, as we have mentioned above, students were reasonably good at expressing their own viewpoints regarding the topic at hand, but did not appear to focus primarily on the given statement. The context of the statement was their focus. Furthermore, if a student had decided a statement was true, he or she was often unable, unwilling, or just resistant to discuss what it would take for the statement to be false. We view this as an indication of partial understanding, much the same way that in order for one to understand the notions of *chocolate* and *linear*, one needs to understand the notion of *non-chocolate* and *non-linear* as well. A second reason why we think students' understanding of the English statements is not so strong is that the students seemed to be unaware of the (possible) ambiguities of the natural language. The statements that we consider to be of EA form were, in many cases, regarded as unambiguously AE. When faced with the possibility that it might be otherwise, students did not entertain the thought and did not go back to re-read the statement in question. They either continued to talk about their viewpoints or they left the ambiguity unresolved. We believe all this is evidence of only partial understanding of the given statements.

3.3.2 Statements in a mathematical context.

The performance dropped sharply when it came to the mathematical situations, Statements 10 and 11. Using the conventions of mathematics to assess the students answers, we required that the student interpret each statement correctly (Statement 10 as AE and Statement 11 as EA). Thus if the student was thinking of real numbers, he or she would have to provide a valid reason why Statement 10 is true and Statement 11 is false. On the other hand, if a student was thinking of integers only (as many students were), he or she would have to provide a valid reason why Statement 10 is false and Statement 11 is true. As it turned out, there were very few students who were thinking only of integers as 'numbers' and got Statement 10 or Statement 11 right. In other words, based on our data, it is not the density of the real line that caused the apparent difficulty students had with these statements.

In addition, there were three students who indicated some confusion about 0 and considered that it could be considered to be a positive number. In two of these cases, because of other aspects of their response, our assessment would be same whether or not this error was reflected. In the other case, this was the only error the student made and we consider the argument to be invalid.

Table 1 of Appendix B shows the percent of students who gave a valid argument for each of the eleven statements (see % valid). In the case of Statements 10 and 11, this refers to those students who interpreted 'number' as 'real number'. If we consider those who interpreted 'number' as 'integer', the % valid goes up from 41% to 50% in the case of Statement 10, and it goes from 9% to 13% in the case of Statement 11 (see table 5 of Appendix B). Since the success rate with these statements was not substantially improved when looking at integers, we chose to list the success rate of those who made the 'real number' interpretation. All the remaining calculations were done with this assumption. A total of 57% were unsuccessful for both statements. The percentage of students successful on Statement 10 but not Statement 11 was 33%, successful on Statement 11 but not Statement 10 was 2% and successful on both was only 8%. (See Appendix B, Table 5).

The difficulties students had with the statements in a mathematical context appeared both in their responses on the questionnaire and were maintained strongly in the interviews. We can provide a number of examples of this.

Here are some questionnaire responses given for Statement 10:

- *Eventually a will be small enough that b will have to be a nonpositive number, making the statement false.*
- *True, if zero is considered a positive number.*
- *False: $a = b$.*
- *True: b can be a number less than a.*
- *False: b might be a larger positive value than a.*
- *False. If $a = 1$ then $b = 0$. Zero is neither positive nor negative.*
- *False. $a = 2$, $b = 1$, $1 < 2$.*
- *False. If $a = 1$ and $b = 0$. Zero isn't a positive or negative number, it is itself.*

Here are some questionnaire responses given for Statement 11:

- *Statement 11 is a reworded equivalent of statement 10.*
- *True: b can be less than a in all situations.*
- *This is the same thing as 10, just rearranged.*
- *True. Same reasoning as in 10: numbers grow and shrink to infinity.*
- *Same statement as 10, just syntax is different.*
- *Same logic as 10.*

- *True. If b is 1, then a will always be greater.*

Now, for the interviews, we begin with some positive responses. During the interviews, several students showed they had a good understanding of at least some of the mathematical statements. Here, for example, HAR came to the interview with a good understanding of Statement 10:

> **I:** OK, let's move on to number 10.
>
> **HAR:** OK, I have to read this one again.
>
> **I:** Yeah, go ahead.
>
> **HAR:** ...(LP)... OK.
>
> **I:** OK. Tell me what you were thinking. Why do you think it's true?
>
> **HAR:** Because for any number a, there will always be a number that is smaller than that. ... By.. by... there are an infinite number, an infinite amount of numbers, and no matter what number you take, however small it is, there will always be a number smaller.

Here is another example, from the interview with SOU:

> **I:** OK, good. Let's move on to number 10. You had said that that's true. Can you tell me why?
>
> **SOU:** ...(LP)... (VLP)... because if you pick one ... a is a big positive number... and there is always going to be a positive number that is smaller... all the way down to if you pick a small fraction, there is always going to be a fraction that is smaller than this, but yet bigger than zero.
>
> **I:** OK.
>
> **SOU:** I think....
>
> **I:** OK, it makes sense.

Here is an example from the interview with LER, who started out thinking that Statements 10 and 11 were the same but, during the course of the conversation with the interviewer, changed her mind about Statement 11.

> **LER:** ...(P)... I think this is clearer ... number 10 is clearer than number 11, because ... it's ... I don't know... [reads it mumbling].. this kind of makes it so ... there is a number which is less than the original... –this is hard to explain – this makes more sense..
>
> **I:** Number 10?
>
> **LER:** Yes. Because ... You say there is a number that has this characteristic. And then here you are saying there is a number who.. for the other... wait... (P) Oh no... it would make more sense to write this as $a > b$.
>
> **I:** $a > b$?
>
> **LER:** Because the last one... like on here, you say b last, and then you define a characteristic of b. And here you say a last, and then you still define a characteristic of b.
>
> **I:** For number 11?
>
> **LER:** yes.
>
> **I:** I see. So what if we change number 11 to say 'there is a positive number b such that for every positive number a, $a > b$'?
>
> **LER:** ...(LP)... That would... see they both make sense....

I: Hm-hm, sure.

LER: And ... ahm...

I: But would that change your opinion about the statements being the same or different?

LER: I think they would still be the same... [reads it again]... (LP)...

I: Ahm, so would you say that it's true – number 11 is true, if we rephrase it that way?

LER: Hm-hm.

I: OK.. Can we find that number b, if that's true?

LER: There is a number b such that for every a, $a > b$?

I: Hm-hm.

LER: ....(P)... OK, well then it would still be the same thing because you would be saying ... so you would start off with... so you can still lose. So it's still kind of ... it's the same as that one. As this was before here. I don't know why this is... this makes ... this works out, and I can't really figure out why.

I: Number 10 works out?

LER: Hm-hm.. So if you change it, there is a positive number a ... such that for every positive number b ... $a > b$, would that work?... I guess it's just what order you are using...

I: So, if it is true, that there is a number b such that for every a, a is bigger than b, can we find that b?

LER: There is a number b ... for every... OK can you say that again?

I: Sure. There is a number b such that that for every positive a, b... excuse me, a is greater than b.

LER: ...(LP).... No. ... The word every is throwing me off there. There is a number b such that for every a, $a > b$... It's the word every and I can't explain why this is... (P) there is a number ... (P)...

I: Take your time.

LER: OK! ... (LP)... NO, that's not true, because... OK.. Then the b would have to be the smallest number in the world. Because for every a, a would be bigger than b. OK, it's not true! ... So that would be false.

I: That would be false.

LER: Hm-hm.

I: Now this rewording that we did, versus the original number 11, are they different or are they the same?

LER: It's different because... let's see if I can explain... You are defining a as like every positive number ... You are not really... It's different somehow... (P).... This one you name a number a and you always find a b smaller than it – on the original. If you change it around, ... your b has to be determined by the a.

I: I see. OK.

LER: And the other way, your a is determined by the b.

I: OK. OK.. All right! That's it. Thank you very much, you have been very helpful.

Finally, here is MAT, the only example of a student who was capable of understanding both mathematical statements on his own. (He had originally interpreted Statement 11 with multiplication, but after the interviewer clarified the point, the student clearly did not need any prompting to assess the statement correctly.)

I: OK, great. Ahm, let's move on to 10. Here you said that statement is true. Can you explain what you were thinking here?

MAT: (P) ahm, is says for every positive number a, there is a positive number b such that b is less than a. So I took as in ... so obviously working between zero and ... I [...] zero is your fiction point and you can pick any object .. or any number a and no matter how close a is than zero or how far it is from zero there is going to be like a b which is the midpoint between zero and a. So no matter what a is, you can always find a b that is always less than that... [mumbles]

I: OK, I would just like to say this for the tape – that as you were saying that, you were using your two hands to express a number line, and points on the number line. Now take a look at number 11. You say that one is true as well.

MAT: (P) [mumbles, whispers] (P) Yeah. OK, I think. So I first I took b to be a number between 0 and 1, so it's a fraction, so then, a fraction of any positive number is going to be less than the number itself.

I: OK – Now I don't know that we wrote this down exactly the way you wanted to write it, so let me read this to you using intonation that you may find changes the meaning.

MAT: OK.

I: There is a positive number b such that for every positive number a, b is less than a.

MAT: OH! You put a comma or something between...

I: Right, for instance we put a comma. Would that be true or false?

MAT: [mumbles] (P) I think that would be false. Because if you fixed b, there is a positive number b, which means you fixed b, ... then you can't say every positive number a is going to be greater than b.

I: Greater than b...

MAT: Greater than b. Because there will be some positive a that will be less than b also. ....

I: Could you say one for sure?

MAT: Ah, you can almost go back to 10. And use the midpoint between 0 and b.

I: Why almost?

MAT: Well, I guess you could! (laughs) For sure. Yeah.

We do not have any more examples of students expressing good understanding of Statements 10 or 11 on their own. Several students did make progress, however, during the course of the interview, particularly after the games (see the next section for details).

About half of the students we interviewed made explicit remarks to the effect that Statements 10 and 11 are the same or equivalent (a few students suggested that Statement 11 is harder to understand, but it is not any different than 10). Here is an example of student remarks comparing Statements 10 and 11.

I: OK.. OK, great. Very good. Ahm, last question. Number 11. Let's see... the statement says there is a positive number b such that for every a, b is less than a. And you have decided that that's true. So, can you explain to me why?

CPE: ... (P)... Because I ... I thought it to be the exact same thing as the above. You just switch the... letters around. ...

I: The a and the b?

CPE: Hm-hm. ... Is that right though??? ...(P) (LP) OK. It switches it around... but it's true... (P) because again, you can always find a number that's less than another.

I: OK. Ahm, ... so what is the difference that you see between 10 and 11?

CPE: Well, 10 is less than 11. Eleven would be the a, and 10 would be the b.

I: No, I meant, the statements...

CPE: OH!!! excuse me! OK.... I really don't see a difference at all.

I: Do you see them as reworded, or... ?

CPE: Hm-hm. Yeah.

Here is an example of a student, SOH, who first sees no difference between Statements 10 and 11, but then makes some progress toward understanding Statement 11 better.

SOH: How is this different than that one?

I: That was my question! Do you see them as different? Number 10 and number 11.

SOH: ...(LP)... There... it's possible that they differ.. I am trying to think exactly how, I mean I see that there is a confusion, when you say in number 10 that for every a there is a positive number b, and here you say ... there is a b... well.... they are the same... (P) They are... yeah... they are just placed differently. I mean you could say for every positive a, there is a positive number b ... (LP)... This just... when you put it that way, you might read it and think ... there is a positive number b such that for every positive number a ... It seems to be more like the controlling factor is b since it was said first. And so it makes it look like you have this b, and say you say it's 0.5, and then you, and then you read such that for every positive number a, b is less than a. And you are thinking, well, .. for every positive number a, well, a could be 0.2 instead of 0.5. And that way I mean it gets confusing, because you have like a control factor in ... that usually when you read it you think there is a positive number b, and there is the control. And then you say, such that for every positive number a, b is less than a. Where you might fill in examples, and say that that doesn't really work.

I: OK, you gave an important example. You said what if b is 0.5 ... then what did you say?

SOH: Such that for every positive number a, b is less than a. What if a was 0.2?

I: Yeah.

SOH: So... (P) I guess... they are different! I guess I would have to change that to false!

I: So, would you say that that's false?

SOH: Hm-hm. Yup.

I: And why would you say that's false?

SOH: ... (P).... It's just different, the way you think about it in your head, like b being the constant, and then ... a being... it could be whatever you wanted to be. It could be equal or less than ... so... Now I am looking back at 10. Ahm..... 10 still seems right to me.

3.4 The unknown in math and in non-math situations.

One possible explanation for the drop in performance in going from everyday situations to mathematics is that, perhaps, for many students, the mathematical context simply represents an unknown domain and this increases their difficulty in making sense out of the structure induced by the quantifiers. Aside from the fact that comparing two real numbers for size (which is the mathematical context of Statements 10,

11) should not really be very much of an unfamiliar context for very many students, we tried to find out about this possibility through Statement 7.

Our assumption is that the nature of medieval Greek poems would be, for most of the students in our study, at least as unknown and unfamiliar as the mathematical context. Nevertheless, the students were quite capable of handling the unknown in this situation. Here are some examples of their responses, taken from the questionnaires:

- *There had to be some poems about love, romance, and/or religion.*
- *I have no idea! If I were to guess, I would say it's false; it sounds too stereotypical.*
- *True. That's all the Greeks seemed to do: fight in wars.*
- *I am not so sure, but it would not be surprising if it's true, considering the history of those times, fights, etc.*
- *I honestly don't have enough experience here to make a good call either way. But I can't imagine all Greek life was predominantly war.*

Considering the above responses, it is apparent these students did not know the answer for sure. In addition, their decisions disagree: some thought Statement 7 was true, others that it is false. Yet the common theme in most of the 63 responses we got was that they were able to make guesses that they could support with arguments that sounded reasonable. They were able to present reasonable arguments about an unknown situation. Many students certainly appeared to be able to deal with the structure of the statement.

Yet, when they were faced with presumably unknown facts about numbers in Statement 10 and Statement 11, their ability to provide valid arguments dropped sharply. One might reasonably expect that students who are already in some college-level mathematics class ought to have *some* understanding of the notions of *positive number* and *less than*. Yet the responses we got seem to indicate that outside the context of a specific class, their understanding of these notions is not very deep. But even if we put this aside and accept that their understanding of these notions was far from solid, the students' ability to reason about the unknown in the context of Statement 7 did not transfer to the context of mathematics.

3.5 Attention to quantifiers.

In this section we give some examples regarding the tendency of students to ignore the quantifiers in a statement, including those who made this error only for the mathematical Statements 10, 11 and not for the first 9 statements.

During the interviews, when asked explicitly about quantifiers such as *everyone, for all, at least one, most, a whole lot,* most students expressed their meaning rather well. When viewing quantifiers in isolation, the students did not have difficulties understanding and expressing the differences and similarities between these quantifiers.

However, when students discussed their thinking process for deciding whether a statement was true or not, we had the opportunity to observe the degree to which they paid attention to quantifiers in a statement. We noticed two main trends: those students who did not pay attention to quantifiers in the natural language statements, continued to ignore quantifiers in the mathematics statements; and, some students who had previously paid attention to quantifiers in the natural language statements failed to do so in the mathematics statements. There were no students who paid attention to quantifiers in mathematics but failed to notice them in the natural language statements.

Here is an example from the interview with GIT who seems to ignore the quantifiers in Statement 6:

> **I:** OK, all right. Good. Let's move on to number 6. The magic key. You have decided that that's true. Can you tell me a little bit about how you were thinking about it.
>
> **GIT:** OK... OK. Ahm... Somebody might be sensitive, ... or... OK let's say that somebody might be sensitive, I mean might hold it inside him... So, ... but... there might be a time where he expresses his feelings. So he unlocks his feelings (laughs)
>
> **I:** Hm-hm, that's good.
>
> **GIT:** So, he unlocks his heart. Where he opens to talk about it. So that's the magic key. He opens his heart sometime. And talks about it.
>
> **I:** OK, now when you were talking about someone, was that a particular person you had in mind or were you talking about people in general?
>
> **GIT:** No, people in general.
>
> **I:** OK. Everybody, or some people?
>
> **GIT:** (laughs)... Ahmmmmm, maybe some people, not everybody ... Well, I don't know if everybody is sensitive, or you know has other characteristics...
>
> **I:** Sure.
>
> **GIT:** But I know some that are sensitive. I know from my knowledge...
>
> **I:** Yeah. Hm... I guess what I am asking about is what about sensitive or non-sensitive, what about people in general justifies that this statement given in number 6 is true? You know, because we can talk about some people being sensitive, and some people not being sensitive,
>
> **GIT:** Yes, yeah.
>
> **I:** ...and so on. But what facts do we know that will justify that this statement is true?
>
> **GIT:** ...(P)... Mmmm... well, I think everybody has his own... characteristics... So everybody has something .. you know, inside him... that is important to him. So, yeah. I think everyone does. Some feeling some characteristics that are staying inside.
>
> **I:** Hm-hm, hm-hm.... And those would work as a key? Is that what you are saying? That the magic key, would those be the magic key??
>
> **GIT:** ... (P)... Maybe... I think ... (P)... Yeah... I think everybody has a magic key that can unlock...
>
> **I:** Why, why?
>
> **GIT:** Because everybody ... (P)... OK, ahhhh.... Everybody might feel saaaad, or everybody might feel something, and ... when it comes to a situation, they might unlock

that, and say it out loud. Yeah. So, everybody, I think everybody has... characteristics...[mumbles]

This student now goes on to ignore quantifiers when asked to analyze Statement 10:

I: OK, all right. That makes sense. Let's move on to number 10 real quick.

GIT: OOOOhhhh..... [mumbles] mathematics? (laughs)

I: You have decided that that's false. Can you tell me why?

GIT: Hm.... (P)... OK, ... Can I use an example?

I: Sure.

GIT: If say a is 2, ... and b is again a positive but 3, then 3 is not less than 2.

I: True.

GIT: SO that's why I said false. Because there might be a positive b which cannot be less than a.

I: Right. Now is that what our statement is saying?

GIT: ... for every positive number... [reads it mumbling] ...(P)... OK, it says 'for every positive number a there is "a" positive number b'. So that says that there might be a ... you know a b that would not work, and there might be a b that will work. It's because of the 'a positive number b'. And we know that ... a ... is a positive ... like ... we know that ... OK that's a is every positive number. For every positive number a.

I: So, you are saying there are some b' s that might work, and some b's that won't work. Like the example you gave, where a is 2 and b is 3.

GIT: yes. Yes.

I: And that's not going to work.

GIT: Yes.

I: Is there a b that would work?

GIT: yeah. Let's say a is 2 again, b is 1.5. So the 1.5 is less than 2.

I: OK. So we have a b that will work and a b that won't.

GIT: Yes.

I: Does this help with the statement ? To decide if it's true or false?

GIT: ...(P)... Well, I guess there might be a positive number a ... which will be less than b. Like it will not work for both cases. You know, if you have an a ... which is greater than b, then this will not work, because we say that b has to be less than a.

I: Hm–hm...

GIT: So, there might be a number for a too that will not work.

I: OK.... So, ... you are saying for an a there isn't a b that will work ...? I guess I am asking you to rephrase, I am a little confused.

GIT: Yeah. There might be an a for which... you know, let's say a is.. an example again, let's say a is like 2, and b is ... no. Let's say a is 3. And b is ... (P)... b... (P)... OK, a is 3 and b is.... (laughs)...

I: Take your time...

GIT: Let me see....(P)... One. or I guess... (LP) OK, a is 3 and b is 4.

I: a is 3 and b is 4, hm-hm.

GIT: So,... 4 is not less than 3.

I: Right.

GIT: Yeah.... (P)...

I: OK, and that will make the statement false?

GIT: Hm-hm.

Here are two examples of two other students who, though they had been paying attention to quantifiers in Statements 1 through 9, appeared to ignore them in the mathematics statements.

I: You seem a little bit uncertain about the [??] but that's OK. [the sound quality is generally bad here] Ahm, what I want to do now is take a look at number 10. I hope you can read the copy there, it's pretty light. Oh, I see, you circled both true and false. On my copy I can't read what you said there, can you read it there?

ERI: It says that it is possible, but not always true. And could be true or..

I: OK. So tell me your thinking there.

ERI: (P) It's possible for you to put 5 for a and 4 for b, and that means that a is larger than b. But it is possible it could be the other way. You could also put in a smaller number. For a. And a larger number for b, and that could be correct.

I: That would be correct? If you put a larger number for a and a smaller for b? But you circled both true and false.

ERI: (LP) That's because it is not always true. (LP) [mumbles]... I don't know... (LP) If you put in 4 for a and 5 for b... or a smaller number...

I: What about all these numbers: for every positive, there is a positive... do they mean anything in this sentence?

ERI: (LP) I don't know.... It's possible... It depends what numbers you put in. It's possible. I think it's true that it's possible, but not always, depending on what numbers you put in.

Finally, we give AND's responses regarding Statement 10:

I: OK. Good. OK let's move on to number 10. Now this statement you say is false.

AND: (LP) Well, cause I mean it depends on where they are located. On the number line. I mean yeah, I mean I know in math they say a is less than b or b is less than a ... but I mean here you are saying for every positive number a there is a positive number b such that b is less than a ?

I: Hm-hm.

AND: Not necessarily true. It depends on where those are located at the point. I mean who gets to pick where a and b are? I mean, I always put a less than b, so to me, I mean if I chose the points, I would pick a to be 1, and b to be 2. So it depends on who is picking the points or if you are saying ... you know ... this is where b is, this is where a is. Therefore, you know...

I: What about the words 'for every' and 'there is' – do they affect anything?

AND: (LP) Well I suppose... because I mean if you were to say for a positive number a there is a positive number b such that b is less than a, but for 'every' ... I mean you are saying in all cases, there is no exceptions to this rule. In every single case. I mean and in some cases, I mean it just depends on who picks where a and b go.

3.6 Difficulties with changing the truth value.

Another indication that students had trouble, even with the English language quantified statements, was the difficulties they exhibited in trying to change the truth values of the statements (which is related to but not the same as negating statements). Many students were unable to give any reasonable responses in the interviews regarding the possibility of keeping a statement unchanged but imagining a situation (or a different world view) in which the statement would have the opposite truth value. Others exhibited a tendency to exaggerate the 'negation'. That is, in order to make a statement false, the student constructed a world or a statement in which the statement is false for every instance of each variable.

We begin with an example of the latter. In constructing a world in which Statement 1 is true, CAR, who had interpreted it as an AE, goes all the way to a world in which everybody hates everybody.

> **I:** I see. OK. OK, very good. Now let's go back to the original statement. "Everybody hates somebody". You decided for your reasons that that's false. OK, now, ... can you imagine a situation where you would decide, with the same beliefs that you have, a different situation where you would decide that this statement is true. Can you describe such a situation? What would the world have to look like in order for that to happen?
>
> **CAR:** OK, everyone hates somebody. When this statement would be true would be like when like ... I see like, ... brown ... like everything being really drab and like no one like smiles anymore, and there is maybe there is it is like nuclear winter or something like that and I see it like I mean you know like they talked about like Europe after W.W. II, I see that where like everyone hates someone.. I mean like where after a period where there's been just mass hysteria and everyone is confused and no one wants to talk to anyone else about anything at all. Not even to complain about how much they hate the world. They don't even want to communicate. So, I mean I see it as just like a time when the earth was like ... nothing was alive... not even , I mean when humans were alive, but their minds weren't, they weren't like really functioning, they were just like sort of walking aimlessly...

This also occurred with statements that the student had interpreted as an EA as in the following discussion of Statement 3 which this student had interpreted as an EA and evaluated to be false. In this case, CPE first gives the negation (a situation in which Statement 3 is true) as essentially an AA, but then responds to the prompt by reducing to a single person as the statement requires.

> **I:** OK. All right. Now once again, just like we did for number 1, is it possible for you to imagine a different situation in which you would decide that this very same statement is true? What would it take for it to be true?
>
> **CPE:** ... (P) ... We have to be on another planet I think! (both laugh) I just can't... I can't imagine everyone being sincere, considerate, kind... all the time, for every moment in their life.
>
> **I:** OK. Ahm, so in that weird planet, would everyone have to be kind and considerate? Or would some people have to be kind and considerate? For the statement to be true.
>
> **CPE:** (P) Just someone. Just one person. Would have to be kind and considerate for this to be true.

The next excerpt shows HAR, who has decided that Statement 3 is false but seems to have difficulty imagining a world in which this statement would be true. She eventually gets it, and, when asked, she show she understands the syntax of the statement.

I: OK, all right. Let me ask you something else. Is it possible for you to describe a situation – imaginary, right, because you have decided that that's false – is it possible for you to imagine a situation where you would decide that this is true? What would it take for that to happen?

HAR: ...(LP).... I am not sure... (P)... I don't know, I don't know. Because again, I relate it to all the time. Being kind and considerate all the time. So... I mean it would take ... I guess my situation would be a case where you around every other people for, you know, a day, or... something like that, I mean I think that it's possible to be kind and considerate to everybody for short periods of time. But when you extend that... I don't know if I am answering your question...

I: You are getting towards that question... Here is what I am trying to focus on. You are saying 'you can be blah blah blah'. The 'you' there is generic, we are talking about people in general.

HAR: Right, right.

I: OK. So.... If the statement turned out to be true, for some reason, even though we are saying here it is not the case, but if it turned out to be true, would it have to be true for ... kind of the generic person in society, or would it be true for just one person, ...?

HAR: It would have to be a special person.

I: It would have to be a special person. Would that special person, let's say this person existed, would that be enough of a reason to make the statement true, or would we need more?

HAR: That would make it true.

I: That would?

HAR: Yes.

I: Just one person?

HAR: Yes.

I: Why?

HAR: Because you say someone, and the one means it only takes at least one person.

I: OK.

HAR: There can be more, but if there is at least one person, then this is true.

Here is another example. SOU had decided Statement 3 is true. She seems to have difficulty even understanding what is meant by finding a world in which Statement 3 is false and the interviewer presents an analogy. She seems to understand the analogy but it doesn't help with Statement 3.

I: It would be. OK. Is it possible for you to describe – I know you decided the statement is true – is it possible for you to describe a situation where the statement would be false? the very same statement would be false.

SOU: If you said '*everyone*'? You mean not changing anything?

I: Not changing anything. The statement is the same, but let's imagine a different world, possibly, right? A different society or something like that ... That would make you decide that the statement now is false. What would the world look like?

SOU: You mean like saying ... where if you looked at it like you said that there is one person that's nice to every other single person in the world?

I: Let me give you an example. OK? Let's make the statement ... let's say we have a bowl, and it's full of candy. OK? And there is some chocolate candy and some lemon-flavored candy, and what have you. And I say 'every candy in this bowl is chocolate'. OK? Could be true or could be false, right? How would we decide?

SOU: Well, it would be false because you said that there is lemon- flavored...

I: There is some lemon-flavored candy. OK. Now let's imagine that we stay with the same statement and we have to describe a different bowl of candy in such a way that our statement is true. Can you describe that bowl of candy?

SOU: ...(P)... Well, if it had only chocolates in it and it didn't have any lemon.

I: OK, that would do it right? So the statement can become true in one situation and false in another situation. Even though it doesn't change. Right? The statement itself remains. It's referring to a different bowl of candy.

SOU: Right.

I: So, I am asking you the question here along those lines. So let's leave the statement as it is. Can you imagine a different world, a different society or something like that where this statement would be false?

SOU: It would be an awful place! (both laugh)

I: It would be awful, yeah... true... But what would it be like? What would it take for the statement to be false?

SOU: One person that wasn't nice to anyone.

I: One person that wasn't nice to anyone.

SOU: At least. I mean wasn't nice to at least one person, one other person.

I: OK. Is that the case now? I mean do we have people in society that are not nice to anyone?

SOU: ...(P)... No. (laughs)

I: No. OK.

SOU: I don't think so!

I: OK, OK, fair enough! Very good. All right. Let's move on ...

Finally, here is another student, LER, who had decided Statement 1 is false. When asked to describe what it would take to make this statement true, she was not even willing to entertain the possibility.

I: OK. All right. Let me ask you something else before we move on. Is it possible for you to imagine a situation that would make this original statement true? What would it take for it to be true?

LER: ...(P)... I mean we would have to like live in different universe... Maybe I don't know, the world is taken over by evil or something. I mean just in practical terms, I think there is always going to be someone who doesn't hate people.

I: OK. all right. Let's move on.

3.7 Creativity.

The difficulty seen just above (or at least reluctance, in some cases) would suggest that students have a difficulty building a mental world unless they view it as realistic to begin with. This is very important because one of the skills one needs to have in order to do mathematics is to create new mathematical worlds without judging them in advance as realistic or not. Creativity for the construction of proofs and counterexamples hinges on imagination, the ability and the willingness to create new mathematical worlds. In fact, this is one of the most important skills mathematicians have.

We can report that students' written responses on the questionnaires often contained jokes. Yet they were not dismissing the statements: they provided a valid justification for their responses (True or False) within their joke. Here are some examples:

- A student declared Statement 2 to be False and responded with *This pot does not have a cover.* There was an arrow pointing to a sketch of a leaf next to it.
- Another student declared Statement 6 to be False and responded with *Yeah? Well, where are Kevin's and Mike's keys? Haven't found them yet...*
- A student declared Statement 5 to be False and responded with *Nope, death is a good thing and hasn't ended yet.*

There were many more funny and playful answers to the natural language statements, but none found in the students' responses to the mathematics statements. We are not suggesting that one needs to be humorous to justify Statements 10 or 11 (or any other mathematics statements for that matter). However, we claim that humor is a sign of creativity, that is, we see these humorous approaches as evidence of some creativity in devising rational explanations. This creativity was noticeably absent, however, in the justifications of Statements 10 and 11. We take this as further evidence that students do not have a comfortable level of understanding of numbers or of a mathematical world.

4 The Game

In the previous section we described those results of this study which had to do mainly with the knowledge that the students had when they completed the questionnaire and what they brought to the interviews. In the latter there was a considerable amount of probing and some prompting, and our main concern was to find out about the mental constructions the students may have been using to make sense out of the 11 statements on the questionnaire.

In the latter part of the interviews, the interviewer made an intervention in the form of asking the student to play a certain game involving first Statement 9 and then Statements 10 and 11. In our mind, this game expresses the logical statement we are trying to help a student understand. Moreover, our use of it is actually a pedagogical tool introduced in our investigation of student understanding. In other words, instead of just asking what the student understands and is capable of doing, we are asking what the student is ready to learn given a particular pedagogical treatment, such as playing a certain game. This could be related to the idea of a "zone of proximal development" put forward by Vygotsky (1986).

The notion of a game is not at all uncommon in mathematics. In fact, one frequently describes concepts of set theory in terms of games with two players. An example of this is the Axiom of Determinacy, which is an alternative to the Axiom of Choice (Moschovakis, 1980). Examples abound in logic, such as Ehrenfeucht games (H.D. Ebbinhaus, et al., 1984), and in computer science (for example in A. Nerode, et al., 1992, the authors discuss how to interpret programs as game strategies).

Furthermore, the notion of a game based on a statement is not new as a pedagogical tool. For example, TARSKI'S WORLD, a software program designed to introduce students to the language of first-order logic uses games extensively. The program gives a student the opportunity to play an interactive game with the computer in order to lead the student step-by-step to mistakes (in case there are mistakes). The games used in TARSKI'S WORLD are essentially identical to the games we described to the students we interviewed. Finally, as another example, we mention a book, (Courant and Robbins, 1996, pp. 311—312) where the authors express the precise definition of continuity in terms of a contract in which one person agrees to produce a δ with a certain property whenever the other contractee produces an ϵ. We have known colleagues who sometimes employ the use of games similar to what we will describe below in order to help students understand a statement. So, although we do not claim to be pioneering this idea, we are not aware of any studies of the use of such games as a pedagogical tool. Therefore we decided to incorporate the use of these games in our present study, by way of a first step in a systematic investigation of a potentially useful pedagogical strategy.

Our specific goals for using a game in this study are to specify precisely what we mean by a game and the conditions under which it can be used; to produce some first data on student reactions to a game; to see what results might be possible from students playing a game in an interview; and to generate tentative conclusions and conjectures for future study of the effects of a game.

We believe that the game has potential for helping those who have some understanding of the context/world that a statement is referring to. For example, we do not think that a game approach for Statements 10 and 11 would help a student who has no (or very little) understanding of numbers to begin with. We consider it to be promising for those who have some understanding of the context of a statement but cannot deal with the syntax to notice quantifiers, the order of quantifiers, etc., in particular for students who are unable to use the syntax to understand the context of a statement.

4.1 The idea of a game

Students were asked to play a game in connection with Statements 9, 10, 11 based on the statement at hand. By 'game' we mean exactly what is meant in the usual sense of the word in the English language. For example, 'game' is defined in the Oxford English Dictionary to be "a contest played according to rules". The game we have in mind in this context is much like ordinary games our students know (such as chess or tic-tac-toe) except that our games are based on some given formal (or formalizable) sentence with one existential and one universal quantifier.

So, suppose we have some sentence $\mathcal{S}$. In order to give a general description of the game based on $\mathcal{S}$, we will assume that $\mathcal{S}$ is expressed formally (this is only for the convenience of the reader so that our description is uniform; once the game is understood, it will be clear that it is not necessary to formalize sentences in order to play their corresponding games). To describe this game based on a formal sentence, let us take a generic example of such a sentence to be

$$(\forall x \in B)(\exists y \in C)\ R(x, y).$$

The game has two players, the A-player and the E-player. The A-player chooses x's from the set B, while the E-player chooses y's from the set C. Who goes first is decided by the order of quantifiers in the sentence. So, in this case, the A-player will start the game by choosing an x from the set B. Then, the E-player will respond by choosing a y in the set C. The players go on like this, alternating. The E-player is trying to establish that the formal sentence is true. So, given some x_0 that was named by the A-player, the E-player tries to find some y_0 so that $R(x_0, y_0)$ holds. If the E-player can do this, no matter what x the A-player chooses, then the E-player wins. On the other hand, the A-player is trying to antagonize the formal sentence, that is, this player is trying to make the sentence false. The A-player tries to come up with a cleverly chosen x^* such that no matter what y the E-player responds with, $R(x^*, y)$ is false. If the A-player succeeds in finding such an x^*, then he or she wins. So, player A has a winning strategy if and only if the sentence is false.

Similarly, we can describe a game based on the generic formal sentence

$$(\exists y \in C)(\forall x \in B)\ R(x, y).$$

Once again, the game will have the two players: the A-player who chooses x's from the set B, and the E-player who chooses y's from the set C. Once again, the E-player's objective is to establish the sentence, while the A-player's objective is to antagonize, and prove the sentence false. The difference this time is that the E-player starts the game (since the existential quantifier occurs first in the formal sentence). So, the E-player tries to find some y_0 such that no matter what x the A-player will respond with, $R(x, y_0)$ will be true. If the E-player succeeds in finding such a y_0, he wins. On the other hand, once the E-player has named a y_0, the A-player tries to find an x_0 such that $R(x_0, y_0)$ is false. If the A-player succeeds, she wins. So, player A has a winning strategy if and only if the sentence is false.

It is easy to see how to generalize this game based on AE or EA sentences to games based on any other sentences which have at least one existential and at least one universal quantifier. Our study, however, did not have sentences of forms other than EA or AE.

4.2 What we found

Based on the transcripts of interviews of 14 students, we may consider the following questions. Does playing the game affect student understanding of the statements we are dealing with? Under what conditions does this effect seem to occur? What are some effects other than directly on understanding the statements that may be present? Finally, there are a few miscellaneous comments which can be made.

4.2.1 Effects on understanding.

The most common interchanges during the interviews were those in which the student appeared to be helped in understanding the statement as a result of playing the game. In some cases it did not seem that playing the game helped the student to understand. In a few cases it is possible that the student's understanding was reduced as a result of her or his experience with the game.

In the next few paragraphs we give some examples of these observations.

Helping the student's understanding. We begin with an example of a student, SOH, who explains her determination that Statement 9 is false with a general discussion about what we do or do not know and concludes with the explanation that the statement is false because there could be plants for which no fertilizer exists. In other words, for her, the negation of a statement of the form "exists f such that for all p, $G(p, f)$" is a statement of the form, "exists p such that for all f, not $G(p, f)$". This is, of course, a correct negation of the Statement 9 interpreted as an AE statement.

> **I:** Let me see. You have decided that that's false. OK, so can you ...
>
> **SOH:** I guess I stopped being metaphorical here, I guess. Ahm, ... yeah, so, what do you want me to say?
>
> **I:** Just go through the thinking. Why did you decide it's false?
>
> **SOH:** Well... I thought about all plants, and I thought about do we even know some of them, how could we even make fertilizers for them... But I guess there are natural fertilizers also,... so... it might be possible, but ... I guess it's just too unknown for me to say that's true. So, ... I marked false because I wasn't sure, I mean if we don't even know about the plants... and even though, I mean I don't even know much about plants, but there could be plants out there that nothing helps them, you know. I think I am being too realistic about it, you probably didn't mean it that way!!

The interviewer continues to discuss the statement in terms of fertilizers and plants, trying to get the student to be cognizant of the quantifiers. She appears to remain confused, however and talks in terms of a world she is imagining.

> **I:** Now let's assume let's say that we have perfect knowledge about plants and fertilizers .. let's say we have a list of all the world's plants and a list of all the world's fertilizers (whether they are natural or chemical that someone has created). So, we have access to complete information.
>
> **SOH:** OK.
>
> **I:** How would you go about deciding if the statement is true or false? So, you didn't have any suspicions at this point, you could really check for sure.

SOH: I could check? ... I mean wouldn't that be evidence?

I: So, what would it take to check, that's what I mean.

SOH: Testing the fertilizers on the plants?

I: OK. So, what would you do? Just describe what would go first. How would you check?

SOH: Ahm... well, I mean you would check off the ones that you know work. Like that are sold in the stores or whatever. I mean things that people already knew.

I: OK. The ones that ... when you say the 'ones' what do you mean, the fertilizers or the plants?

SOH: The fertilizers ... I mean I don't know ... I mean I just don't know what it means having perfect knowledge because I just don't have it...

I: Let's imagine we do.

SOH: Even if I knew about some of the plants, even if I was all- knowing about plants, I might not be all-knowing about the fertilizers that would go with that plant. Maybe if I knew everything about plants and knew how chemicals would react with other chemicals. But I guess I would have to experiment with all of them. Collect data, and find out...

I: OK. And then what would it take for you to decide — let's say you went through your collection, your data, and all that. What would it take for you to decide that the statement is true?

SOH: I would just have to see it.

This continues through several attempts by the interviewer and the student displays varying degrees of understanding. Then the game is introduced to the student, and after some discussion, she appears to have come to an understanding of Statement 9.

I: OK, great. All right. So, now I want us to look at this data collection and all that, as some information that we have access to – that's what I meant earlier that we have perfect information. Imagine that we have done the experiment, or someone else has, imagine that we have a list of everything. Let's say we are playing a game, the two of us. And we take turns, and I choose a plant, and you get to choose a fertilizer ... And I get to go first. So I choose a plant, then you respond...

SOH: This is like playing the game memory?! (laughs)

I: I don't know that one.

SOH: OK, never mind.

I: I choose a plant, and you choose a fertilizer , and then I choose another plant and you choose another fertilizer , and we go back and forth like this. I try to win, and by this I mean I try to find a plant for which there is no fertilizer , so for which you can't come up with a response that will work for my plant. If I can't win, then you win.

SOH: OK.

I: Now, let's say that you decided that this statement is false. Who gets to win this game?

SOH: You win... no. ... Let me think of the game again! (P) You are trying to give me a plant that I can't find a fertilizer for. And if you win, the statement is false. .... Which is kind of like saying there is a plant with no fertilizer ... Kind of a different statement ... but... hm... I haven't had logic or anything, but it seems to work that way... seems logical to me.

I: OK, so in this case, if you circled 'false', I would get to win this game. If you had circled 'true' , who would get to win this game?

SOH: ...(P)... [mumbles]

I: Let's say the statement here, that there is a fertilizer for all plants was true.

SOH: Is true?

I: Hm-hm. Let's say we decided that that's a true statement.

SOH: And so the object of the game was for you to prove that's true?

I: No, the object of the game is once again: we each are trying to beat each other, right? Win. So, I go first, I choose a plant, then you go and you say a fertilizer , and then I choose a plant, and you choose a fertilizer , and so on. I win if ...

SOH: If you say a plant and I can't name a fertilizer .

I: Right. Otherwise, you win.

SOH: Right.

I: So, let's say the statement here is true. Who wins the game?

SOH: I do.

I: You do. Why?

SOH: Because I would have an answer for you. Every time, if there was really a fertilizer .

I: OK. Good. So, now let's do a variation on this game. And the difference is... you still get to choose a f and I still get to choose a plant, but we reverse the order. You go first: you choose a fertilizer and then I choose a plant. Now,...

SOH: It's not the same, though. Because I would have a fertilizer but there is no plant for it ... so...

I: OK. So let's say I would win in this case if ... once you come up with a fertilizer I can come up with a plant which doesn't like that fertilizer ...

SOH: OH!

I: Otherwise, you get to win.

SOH: That would be easy for you to find a plant that doesn't like that fertilizer because I think a lot of fertilizers can kill plants... (laughs). So, what was the question?

I: So, this game would be won by a different person, right?

SOH: (P)

I: Who would win this game?

SOH: You would. Because you could find a plant that would not like a fertilizer . I don't think there is a universal fertilizer that for every plant would say you help me. You know!

Although SOH came to a better understanding of the statement after playing the game, there was also considerable discussion not related to the game and we cannot be sure that this did not help so the game may not have been the only reason for her improvement.

The following transcript is more clear. HAR begins with the serious error of thinking that Statement 11 is the same as Statement 10. She holds to this idea fairly strongly in discussion with the interviewer, so we might infer that this discussion did not help. Then the interviewer explains the idea of the game, the specific rules, and how it is determined who wins and who loses. After playing the game, HAR not only comes to a good understanding of Statement 11, but spontaneously expresses the view that it was playing the game that led to her understanding. In fact, it does appear that the student's understanding of the game contributed to her understanding of the statement because she revised her answer to agree with her decision about who wins the game, and not the other way around. [The transcript that follows is somewhat long. We are including it here in its full length because we would like the reader to be able to see in detail the transition the student makes as she struggles with her understanding of the statement.]

I: OK...Let's go to the last one, number 11. You have decided that that's true. What is the reason?

HAR: MMM . . . This one confused me . . . (P) . . . I didn't see any difference in Statement 10 and 11, none whatsoever.

I: OK, that was going to be my next question! Do you see any difference?

HAR: No. ... I mean I looked at it, and I looked at it, and I... I don't see any difference.

I: Ahhh, do you see them as slightly reworded, or...

HAR: Yes. Because all you did was you put 'for every number a' up there. I don't see any difference. ...(P)...

I: OK. ...(P)... Let me ask you something else...

HAR: Am I supposed to be seeing a difference?

I: No, I am just interested in just how you see it – exactly. Let me ask you something else. Let's say we play the same, sort of the same game that we were describing before, for number 10. I was doing the a-choosing, right, I would choose the a, and then you would try to find a b and so on, we would go back and forth. Let's say I still do the a-choosing, and you do the b-choosing, but we reverse the order in which we do this. So, you get to go first. OK? So, You find a b, positive , some number, it's up to you to choose it. And then I respond with an a. Ahm, if your b is less than my a, you win, or if we continue like this, you survive, you win just like before. OK? You were trying to make your b's smaller than my a.

HAR: Right, right.

I: So, if your b is smaller than my a, you win.

HAR: But your goal is to make your a bigger than my b? Yes?

I: No. You win if...

HAR: You are changing the rules!! (laughs)

I: No, no (both laugh)

HAR: Yes, you are!! Because you are ... you are changing the way that... you are making it your advantage to win...

I: The only thing I am changing is the order in which we are playing.

HAR: I know, because you have... OK, I ... I choose a number. And but I have to make sure that my number is smaller than your number. You know my number before you say your number.

I: Correct.

HAR: So therefore, you can always pick a number smaller than mine.

I: And thereby, who wins?

HAR: You.

I: I win.

HAR: But that's different! That's a different game than what we were playing in number 10.

I: Oh, absolutely. It is different, yeah. I am not suggesting it is the same game.

HAR: OK, right. Because you know my number I know your number. That's the difference.

I: Yes. So are you sure that I would always win?

HAR: Yes, for the same principle as that... Because basically, when you ... the game that you just did, .. OK. in here, over here, you chose the a and I chose the b.

I: Hm-hm, hm-hm.

HAR: And you said that ... in this one, I 'm gonna choose the b, but I am gonna choose the b first. So basically what you are doing is I... the b becomes the a....

I: OK...

HAR: And your... and what you chose as your a, ... becomes the b up here.

I: I see, hm-hm. OK. Now let me ask you this. Is it is conceivable that number 11 might be describing this game? The second game?

HAR: ... (LP)... Yeah, I guess so... Because it is saying ... because the way the statement is, you choose the b before you choose the a.

I: Where in the statement do you see that?

HAR: Oh, just in the order...

I: In the order ...

HAR: Not what the word says, but ... but once you said that – I wouldn't have seen it...

I: Sure...

HAR: But once you said that, I can see the statement is telling you to choose b before you choose a. And so, because of that ... I don't... ahhhh... see... OK, me saying that you would win, on this, the game we just played, is based on the fact that I ... the assumption that you want to win. Do you see what I am saying? That you will manipulate the game so that you can win.

I: Hm-hm.

HAR: Because .. OK, you said that I would choose a b. And I could win if you chose an a that was larger than my b.

I: Correct.

HAR: But I am making the assumption that you don't want me to win.

I: True.

HAR: So therefore, you will choose a number smaller than mine.

I: Sure.

HAR: That is not implicit... or that is not in the statement ... I mean, because you change it into a game, I saw it as a win or lose, and manipulation... And here, I don't see it the same way. I don't... Because I do believe .. I mean there is an a that is larger than b. So this statement is still true.

I: 11 is still true?

HAR: Hm-hm [affirmative]. Even because ... if you choose a b ... you can still always choose an a that's larger. So this statement is still true. But the reason that I said that you would win, is because with the game, I made the assumption that you would want to win.

I: Sure.

HAR: And that's not in this statement ...

I: OK, I understand. Yeah, the assumption with all these games is that each player is trying desperately to win.

HAR: Exactly. And that ... that doesn't... I mean because this is just a blank statement ... I don't see it as a game. You know?

I: OK. I understand, yeah, yeah. It makes sense. Because there is no game and so on..

HAR: Right.

I: Ahm, but you said, you said something interesting. You said for the b that you may choose, there is going to be a's that will be bigger. Right? Which will help you win. But there will be a's that are going to be smaller.

HAR: Yes.

I: Ahm, what is the ... reference here to [aim?] the statement number 11. Does.. how does it qualify the a's?

HAR: [mumbles]...for every positive ... (P) ... (LP)... AHHHHHHHHHHHHH [in pain] (laughs) ... You are confusing me!!! (both laugh) ... (P)... OK. I see what you are saying. Because by s... I am being contradictory, because if what I said is every... because a ... that if I can also choose an a that is less than b, then the 'every' doesn't work...

I: Hm-hm.

HAR: OK. So, it is a different statement .

I: yeah... would you say it's true or false? Number 11.

HAR: ...(P)... False.

I: False?

HAR: Hm-hm...

I: Why?

HAR: Because you.. you forced me to see that you choose b first... And then, and then you choose a. And, whatever .. once you choose b, ... you can choose a number that's larger than b or a number that's smaller than b. But in 10 it's different, because you say 'every number a' and then you say "a" positive number b, and so that makes it go the other way.

I: Hm-hm. Ahm, now the reality is, of course, that it's like you said. You see no game. There is a statement and it's about numbers, you don't see a game. So is it possible to decide and answer like 11 is false, like you said, without reference to a game?

HAR: Without you.. I wouldn't have noticed the difference without you saying so.

I: Oh, I understand that, that makes sense. But now that we have established that there is some difference,

HAR: Hm-hm.

I: .. is it possible to justify that number 11 is false without saying hey, we can talk about a game, and so on. Can we just talk about numbers and no games, no players, and still justify that number 11 is false?

HAR: You mean come up with specific examples?

I: Ahm, no, just kind of argue why number 11 is false. What's wrong with it?

HAR: And not reference it to a game?

I: Right.

HAR: ...(P)... Yeah, I think it's possible, I mean I think it ... I think takes someone knowing ... someone knowing the idea of this game before and helping somebody else through it... (P)...

I: Hm-hm. So, how might the answer go, how would we explain why 11 is false?

HAR: ....(LP)... By saying that you choose one b, and I see it as like the span, you know, like a number line...

I: Hm-hm.

HAR: You have a b and for every, every number, means every single number would fall over here ... that a, every single number a would fall over here. And that's not true.

I: To the right of b?

HAR: Yeah. Because there are things that will be on the left of b. There are infinitely many things over here.

Having no effect on the student's understanding. It is not always clear that it is the game that helps a student increase her or his understanding. The following excerpt is from a student, SAN, who said on the questionnaire that Statement 11 is true. Here she changes her mind, gives a good explanation of Statement 11 and seems to understand that it is different from Statement 10.

I: OK, great. Ahm... number 11.

SAN: ...(LP)... (LP)... That's false!

I: Number 11? You can revise your answer, yeah, that's fine.

SAN: Well, let me read it one more time.

I: Yeah, yeah, take your time.

SAN: ...(LP)... It's false.

I: Why is it false?

SAN: Because... you are saying that there is a positive number b that you choose ... that for every for any choice of a will... for every positive number a ... b is less than a. But ... based on the statement that we said before, there is always a number ... ahm... (P).. You are always going to be able to find an a that is less than your choice of b.... like we said before, there is an infinite number of numbers between ... the number b that you choose ... and any other number including zero, so you would always be able to find an a that was less than that b.

I: OK, so you would say that that's false? OK, that makes good sense. Let me ask you what you have written here. Which was 'see above'. At the time, if you remember, were you thinking that these were the same statement ?

SAN: Yes, hm-hm.

I: OK. What changed your mind now?

SAN: Hm.... (P)... I guess the first time I didn't ... I didn't realize exactly what the statement was saying... and... now that I read it again,... I see that it's saying something different...

I: Hm-hm. What is the difference that gives it away?

SAN: ...(P)... It is a different question... almost the opposite question than the question before.

All of this discussion takes place with no mention of the game relative to Statement 11. The interchange occurs at the end of the interview in which the game has been used for other statements, in particular for Statement 10 which was discussed just before this excerpt occurred and where it does appear that the game helped her understand that statement. Thus it could be that the effect of the game carries over, or it could be that for this statement the game is not having much effect. One indication of the latter possibility is that the student indicates that it was reading the statement again that made the difference. On the other hand, when it is all over, the interviewer brings up the game and the student expresses her opinion that the game helped.

> **I:** So the order in which we play is different. Making me the winner and making this statement false. Whereas in the first one, you were the winner and the statement is true. I guess the reason why I am describing in terms of a game is because I am curious to see whether it is helpful.
>
> **SAN:** It is.
>
> **I:** Do you think that it helps?
>
> **SAN:** It is helpful when you try to pinpoint *why* it is ... that the ... I mean I can look at the statement and tell you that it's false a lot easier than I can tell you ... I can pinpoint exactly why it is that it is false.

Possibly reducing the student's understanding. Finally, we must ask if it is possible that playing the game can be counterproductive and increase a student's confusion. The following excerpt is from a student, JUL, who is struggling to understand Statement 9. The interviewer prompts her, using the language of the situation and she appears to be making some progress.

> **I:** One possibility then would be that there would be one fertilizer for all plants...
>
> **JUL:** Hm-hm.
>
> **I:** What's the other... what's another possibility?
>
> **JUL:** That's what I didn't know. Yeah.
>
> **I:** OK. So, is that the interpretation you gave it?
>
> **JUL:** That there was... that there was one fertilizer for all plants?
>
> **I:** OK...
>
> **JUL:** Yeah.
>
> **I:** And you thought that was true.
>
> **JUL:** Yeah. (P)
>
> **I:** Can you tell me why that's true?
>
> **JUL:** Ahm... That there's ... not necessarily the same fertilizer for all plants, but there is a certain like kind of fertilizer for all plants.
>
> **I:** But not necessarily the same you say?
>
> **JUL:** Right.
>
> **I:** Oh. So that's not ...
>
> **JUL:** Oh, is that ...
>
> **I:** ...well...
>
> **JUL:** OK, no! I guess... I see a difference between: there is "a" fertilizer and there is one fertilizer ... OK, so now ...
>
> **I:** Right...
>
> **JUL:** OK, so that's what ... I ... if it said 'one' I would say false. Because it might.. that I would take that as the exact same fertilizer for all plants.
>
> **I:** And you don't think that's true.
>
> **JUL:** No.
>
> **I:** Why not?

JUL: Ahm... plants need fertilizers different fertilizers ...

I: different plants need different fertilizers ?

JUL: Yeah.

I: OK. So what about the other way now, the other interpretation?

JUL: There is "a" fertilizer ... That there is ... there is "a" fertilizer not necessarily the same one for all the plants, but there is a fertilizer made for all plants...

Then the interviewer introduces the game. It may be that the student becomes more confused and at the very end, she indicates that, for her, the situation and the game are different.

I: OK. OK.. Let's suppose we play a little game with this one. Now in this game we have to pretend that you and I are both experts on plants and fertilizers – that is, we know everything. [Student agrees throughout] So, here is how the game goes. I go first. OK. Then after I go, you go, and then I go, and then you go, and so forth as long as it keeps going. And what I do is I pick a particular plant. OK. Then your job is to find a particular fertilizer for that plant. OK? If you can't, then I win. If you can, then I go on, and name another plant, and you have to name a fertilizer for that, OK? So, we continue in this way. If we ever come to a time where you can't name a fertilizer then I win the game. If that never happens, and we continue doing that, then you win the game. You understand the game? [Student agrees throughout] OK. Who wins the game?

JUL: (laughs) (P) Ahm... I would probably say that you would win. Because I think that you could probably name ... since I don't really ... I think there is a lot less fertilizers maybe than there is plants...

I: You are allowed to use a fertilizer twice if it works.

JUL: (P) ahm.... I would still say that probably ... you would win... well... yeah, I think it'd probably be easier to name plants than fertilizers ...

I: So what would happen at some point is that I would name a plant, and you would...

JUL: ... Unless it was the same... (P) I mean there are probably some plants that yeah would take the same fertilizer .. so..

I: Right. You are allowed to do that. But you still think it's false.

JUL: Yeah.

I: And that means that there would come a time when I would name a plant and you couldn't name a fertilizer ...

JUL: Hm-hm.

I: But before you said that the statement 'there is 'a' fertilizer for all plants', you said that was true. Now what's that got to do with the game?

JUL: Then ... because if you can name a plant then I can't name a fertilizer ... if you ...if you win the game by naming a plant, I can't name a fertilizer ... Then I am saying that there is not a fertilizer for all plants.

I: So that statement would be ...

JUL: False.

I: But you said it was true before.

JUL: Yeah...

I: OK. Which way do you think now? True or false?

JUL: (sighs...) ...

I: I know this can be disconcerting..

> **JUL:** I know... ahm (P) I guess now that I ... if I think of it in that way, like the game then it would be false. But... (P) ahm, but still... see I guess I see it ... (P)... I guess I see how it's the same thing how you are saying it's the same thing but yet when I look at that statement, it means something different than the game, I guess.

In this case, the student did not have a very good understanding of Statement 9 and it seems clear that the game did not help. It is possible, although we cannot be sure, that playing the game increased the student's confusion.

4.2.2 Conditions for the effects.

Thus we see that playing the game, like every pedagogical strategy, can have different kinds of effects for different students in different situations. There is need for careful studies to determine, as seems to be the case here, whether playing the game is more often helpful than neutral, and to make sure that it is not harmful. Another point of view is that, given the fact that playing the game can help the student understand, it is important to investigate the conditions under which it is most likely to be helpful.

To begin such an investigation, we offer the observation, which these interviews generally seem to support, that in order for the game to help, the student must understand the rules of the game and know exactly how it is determined who wins and who loses. If the student is not given this information, or if (as occurred in some cases) the student is asked to devise the game, then understanding may not be increased. This is not surprising. Our point is that the game may be an effective device for the interviewer to help the student construct an understanding that fits with the interviewer's understanding. This understanding is expressed in the game, so it is reasonable to insist that the game come from the interviewer, at least at first.

In the following excerpt from AND, the interviewer fully explains the game for an AE interpretation of Statement 9 and the student seems to understand the statement. Then, the interviewer asks the student to change the game (to get to an EA interpretation) but does not specify the conditions under which each player wins. The student's response indicates an unwillingness to accept an EA interpretation of the statement. This data is consistent with our suggestion of a strong connection between fully understanding the game and interpreting the corresponding statement.

> **I:** So then, ahm, would in your interpretation, is the statement true? 'there is a fertilizer for all plants' ? [mumbles] You still think it's true. OK. Let's play a little game with this statement, OK? And the game is the following. I go first, and first of all when we play this game, we will assume that you and I know everything there is to know about fertilizers and plants. OK, we know all the answers. So I go first and I pick a particular plant, and then your job ... you have to pick a fertilizer that's a good fertilizer for it. And then if you can't, I win.
>
> **AND:** If I can't?
>
> **I:** If you can't I win the game. If you can, then I get to pick another plant, and you try a different fertilizer . And if you cannot pick a fertilizer for this one, then I win. But if you can, then we go on. If ever I pick a plant that you can't pick a fertilizer , then I win the game. But if that never happens, then you win the game. OK, you understand the game?

AND: OK.

I: Who wins the game?

AND: (P) Me.

I: You would win this game. Because?

AND: because all these nutrients in the soil, there is always going to be some type of fertilizer there as well as God to help it, too.

I: OK, now I don't like that game, since you win so I am going to change the game, OK, see if I get a better chance, OK? This time, you go first. You pick the fertilizer and I have to pick a plant. Does that change anything? Who wins now?

AND: You will, because no matter what plant I pick, you can do the same the thing I would have done because there is always nutrients in the ground and God's help to make it grow. So therefore you'd win because whatever I would say, you'd be able to come up with some type of plant that could use God's help or could use those nutrients.

4.2.3 Other effects.

There were other effects that indicate that discussing the game can be a meaningful experience for students and that it can contribute to the learning/teaching enterprise in several ways. These include the fact that the students tend to adopt the language of the game, and that this can lead to improvements in the clarity of their explanations.

Adopting the language of the game. LER provides an example of the many cases we saw in which students used the language of the game to discuss the statement. Here, in finally realizing that Statements 10, 11 are different, she uses language in which two people are active in choosing and naming numbers. She did not speak this way before playing the game.

I: Now this rewording that we did, versus the original number 11, are they different or are they the same?

LER: It's different because... let's see if I can explain... You are defining a as like every positive number ... You are not really... It's different somehow... (P).... This one you name a number a and you always find a b smaller than it – on the original. If you change it around, ... your b has to be determined by the a.

I: I see. OK.

LER: And the other way, your a is determined by the b.

The use of game-like language indicates that the student understands the connection between the statement and the game. This is very likely an important step in developing an understanding of the statement. Eventually, of course, we would like the student to be able to analyze the statement without explicit use of artifices such as the game.

Improving the clarity of explanations. Aside from indications we have seen in which the game appears to help the student directly to understand the statement, we saw examples in which the understanding may have been fine before playing the game so there was little room for improvement. Nevertheless, the student's explanations in some cases became richer and clearer. Compare, in the following excerpt, the explanation of CAR for Statement 10 before and after she plays the game.

I: OK, great. Now I want us to go to the next question, number 10. You have decided that that's true. Why?

CAR: OK. (P) They are both positive numbers, and a is always greater than b and so, the way I am looking at it, is that a can always... is ... I mean since like you have more numbers than you can ever imagine ... and so, you can just keep going up and up and up and up with a, and then since it says right here b is less than a, then b you know, always has to be a number below it. It can be below it by like 10, or by 1000, and so it is just going to go on forever.... I mean the sentence just makes it true. It doesn't say 'less than' ... I mean it doesn't say 'equal to'. So it's just gonna... it's just plain to me that a is always gonna be bigger than b. Because you have so many numbers.

I: OK. So, let's say we are trying to cook up a game just like the one we had with the fertilizers . Let's say that we are trying to make up a game that goes with this sentence. Can you imagine how might that go? It's the two of us playing the game.

CAR: OK. You'd say....(P)

I: Who goes first, what do we choose... ?

CAR: I'd say I'm gonna go first, and I am going to give you a number a, and you give me a number b that's less than that. And I would always like, you'd say, like, no matter what number you said, I could always give you a number less than that. Well, wait a minute... except... if you gave me the number zero, because then the only number left would be a negative, and the question says positive .

I: And could I give you a zero?

CAR: Ahm, (P)... I think so, I mean, it doesn't say that you can't give me a zero... But then, zero is... I don't know I have always had trouble with zero. Like ... it's a weird number, because you wonder: is it positive or is it negative? And... and then you think about ... like what zero can do to a number, once you put a zero by it, just like double it or triple it, or make it like go crazy, and so... zero confused me because I don't know if it's positive or negative. So, I think of it almost like a neuter or something.

I: It is actually. Zero is neither positive nor negative.

CAR: OK. So, then you couldn't give me zero.

I: OK. So, let me make sure we have the game straight. Who gets to go first?

CAR: You go first.

I: I go first and I choose an a?

CAR: Yeah. And I always give you a number that's less than that.

I: OK. And so how would we describe who wins?

CAR: Ahm, I'd win. Because I can always go below you. (P) No, no, no... I could go below you to a point... and I can go below you until you choose 2. Because the minute you choose 2,... no, because I could choose 1. OK you choose 1, and then it's sort of like a draw. Because I can't choose zero because zero is not positive , zero is neither. So I guess we could say that you won that way, but I don't think it would be fair. (both laugh)

I: So, if I were to choose 1, and make it difficult for you, what would it take for you to win? You would have to come up with what?

CAR: Well, if you chose 1, I could into like decimals. I could say 0.5, but then you could go like, ... OK, I could think of ... if you kept going down and down and down through the decimals, then we could go, I mean ... we could go for like an infinite amount. We could go to like the point 000009, you know, we could just keep on adding zeros and zeros. I would win!

I: So you would win...

CAR: Yeah!

4.2.4 Final comments.

We end this section with two general comments about the game and its effects. As we have seen elsewhere in this study, students have greater difficulty with EA statements than with AE statements. This might at first appear surprising to those mathematicians and logicians who understand such statements so well that each appears equally simple. However, the difference is profound for students. We find that this difference persists in the context of the game, whether or not the game is helping the student to understand, and we take this persistence as additional evidence for the depth of the difficulties students have with quantified statements.

Finally we comment that in future investigations we would like to study the effects of the game more thoroughly in the following way. A first stage would be where the students are given a full description of the game that arises from a sentence. (They do not make up part of the rules, for example.) There should be interviews in which the student is given a full opportunity to understand the statement before the game is introduced. (In some of the interviews here, the game was introduced fairly quickly and, although it appeared to help substantially, it could be that *any* additional discussion of the statement would have helped at that point.) Gradually, later, they should be asked to contribute part of the game description. Eventually, students get to the point of devising the game entirely on their own.

Though we have not studied this, our point is that this latter is what we envision happening. This study suggests that that students cannot be helped by a game they are asked to devise when they do not understand the statement to begin with, but ultimately we would want them to be able to do this on their own since we believe that one of the true "tests" of understanding is the ability to translate a problem from one setting to an apparently very different setting. If we can get students to do these translations on their own, then we can be more sure of their understanding and the effects of the game. Of course this will not happen in a single interview, it requires significant pedagogical intervention.

5 Discussion

In this section we summarize the conclusions which can be drawn from the results presented and then discuss some pedagogical suggestions that arise from this study.

5.1 Conclusions

Following are the main conclusions that we feel can be drawn about the students in this study.

1. The students did not appear to use mathematical conventions to interpret the questionnaire statements (the mathematical and the non-mathematical statements).

2. The students did not appear to be aware of having engaged in *interpreting* the questionnaire statements. That is, aside from which interpretation conventions they might or might not have used, and aside from their own interpretation (AE or EA), they did not focus on the statements as entities on their own right. During the interviews, when they were told explicitly that the focus was on the statements, and that their opinions were needed solely for establishing the truth value of a given statement, the students still focused on their opinions about the general topic of a statement, rather than how these opinions related to and justified the students' assessments of a statement's truth value. Even when we raised the possibility that a statement might be ambiguous or when we offered an alternate interpretation (such as an EA interpretation when the student's interpretation was AE), in almost all cases the students assessed this possibility by referring to their opinion about the statement's topic, not by referring to the statement itself. This behavior suggests to us that for the most part the students were not conscious of the fact that in one way or another they were interpreting statements, and that they were apparently unaware of any conventions they might have used to make their interpretations.

3. Continuing on the trend described above, students did not appear to use the syntax of a statement to analyze it. Rather, they referred to a world they were already familiar with and they considered that the statement described that world. They interpreted the statement and decided on its truth or falsity based on the nature and properties of this world. If the statement described a world which was unfamiliar to the student, then he or she was very likely not to exhibit much understanding of the statement. The most important point is that what the student can understand seems to depend on what worlds he or she is able to think of and this scope is not expanded by thinking about the statement.

4. Given an AE statement that is about everyday life, these students seemed to be able to interpret it as such and give a valid argument for its truth or falsity.

5. When faced with a possibly ambiguous statement, students tended to interpret it as an AE statement. Even when the possibility that a statement might be interpreted as EA was brought to their attention, almost all students either dismissed it quickly and without explicit reference to the statement, or they left the possibility unresolved.

6. There was a close relationship between making some interpretation of a statement on the questionnaire as an EA or as an AE statement that was articulated clearly and giving a valid argument for the truth value they assigned to it.

7. Students had great difficulties imagining a situation in which the truth value of a given statement changes. This is the case even for statements which students at first appeared to understand.

8. The ability to interpret correctly and to give a valid argument for truth or falsity dropped significantly for both AE and EA statements when the situation is in a mathematical context.

It seems that the eighth item above points out a *semantic* difficulty on the students' part. Positive numbers are not as real to them as children, fertilizers, pots and covers, etc. Items 5 and 7 suggest that even with statements about situations familiar to students, their understanding of multiple quantification is not very strong. Thus, even if we were able to effect a transfer from everyday to mathematical contexts, the knowledge we are transferring may not be as strong as we might have hoped. But Item 3 above suggests it might be over-optimistic to expect that very much of this transfer will take place. Items 3 and 4 suggest that there is a *syntactic* difficulty in the sense that most students failed to make use of, and/or were unable to understand the structure of the statements. Moreover, they had difficulty viewing a statement as an entity on its own right, apart from any truth value, and apart from what it means in some particular world.

If indeed it is the case, as seems to be suggested by this study, that students have difficulty viewing a statement as an object in its own right, apart from what the statement 'says' in the (only) world they have in mind, then the implications are many. This would point to reasons why, for example, our students often read (and memorize) a theorem that appears in a box of a textbook, and can only apply it as it is stated in the book. Applying the theorem by using its contrapositive form, for example, often seems to require a 'different' theorem. The above observations could provide some of the reasons why many students have difficulty experimenting with different mathematical 'worlds' in order to assess a mathematical statement, to come up with counterexamples, etc.

Finally, the substantial drop we observed in the students' ability to handle the given mathematics statements when compared to the natural language statements strongly suggests that the conventional wisdom to *teach by making analogies to the real world* can fail dramatically. Our study suggests that simple analogies don't work. Many of these students were able to produce valid arguments to support their answers to Statements 1 through 9 on their own. A major breakdown occurs when they move to the mathematical realm. Furthermore, even regarding the natural language statements, when we probed a little more, their understanding appeared to be more shallow than we first thought by reading the questionnaire responses. Thus, there are two arguments against making analogies to natural language when trying to help students understand mathematical statements. One is that we cannot take advantage of something that may not be there. Second, even if it is there, it is in a form that is not likely to transfer to the mathematical realm. This study would suggest that in order to help our students, we need to remain in the mathematical realm and approach the difficulties from an integrative viewpoint: combining the study of syntax with the study of semantics.

5.2 Pedagogical suggestions.

One of the key issues of this research is the difference between two approaches to making sense out of a complex statement. One is to imagine a world which the statement describes and to reason about that world. The other is to use rules of syntax as an aid in constructing meaning for the statement. Our position is that the latter approach is a powerful one that can be used to construct meaning when the sentence is too complicated to imagine a world directly. On the other hand, it seems that many students are restricted to the former approach and this might reduce the scope of statements they are capable of understanding. Therefore, our most important pedagogical suggestion is to find ways, perhaps including, but certainly not restricted to, the use of formal language, in which to help students learn to use the syntax of a statement as a tool for making sense out of it — in short, our advice is to *Teach the syntax, teach the conventions of mathematical discourse relating to syntax.*

In this section we consider two different approaches one can take in order to integrate the active and explicit study of syntax in the curriculum: computer activities and the game.

5.2.1 Pedagogy related to computer activities

ISETL. One approach has already been used and studied before: the incorporation of a programming language such as ISETL. ISETL is a language whose syntax closely imitates that of mathematics. In asking students to read and write code in ISETL, we ask them to observe, understand, and construct formal statements that express a precise idea. The ISETL approach is introduced and analyzed in the following references: Dubinsky 1995, 1997, Asiala et al., 1997. ISETL is free and can be downloaded from the web. It is available for Unix, IBM-compatible computers (in DOS and Windows versions) as well as Macintosh computers. For more information on ISETL and download sites, look at htp://csis03.muc.edu/isetlw/about.htm.

TARSKI'S WORLD and HYPERPOOF. TARSKI'S WORLD and HYPERPROOF are software packages that were written by Jon Barwise and John Etchemendy, and are designed to teach students various aspects of mathematical syntax, logic, proof, and sound reasoning. They are innovative in part because they involve the rigorous use of visual information, not only symbolic statements. In other words, these packages represent an approach that goes well beyond the more narrow approach of traditional formal logic and the syntax of programming languages. Unlike ISETL, or another programming language, this method skillfully combines syntax and semantics in a natural and even playful way while maintaining a sharp distinction between syntax and semantics. HYPERPROOF in particular extends the traditional notions of syntax, semantics, logical consequence, and proof, in ways that are both rigorous and natural (see Allwein and Barwise, 1996, Barwise and Etchemendy, 1990). These packages were awarded the 1997 Educom Medal for the innovative software. TARSKI'S WORLD is available for IBM-compatibles and Macintoshes, while HYPERPROOF is currently available only for Macintoshes. For more information on these packages, and a paper by Barwise and Etchemendy describing their viewpoints on pedagogy and logic, the reader can look at htp://csli-www.stanford.edu/hp/index.html.

5.2.2 Pedagogy related to the game.

Finally, another approach that appears to have some promise is the use of (mental) games based on a statement with at least two different quantifiers. The use of such games is illustrated in our interviews and what we obtained here can form the basis for designing, implementing and studying a pedagogical strategy in which such games play a significant role. This is a matter for future investigations.

In light of the fact that the pedagogical tools we mentioned above can provide substantial help for students with the syntax (and correct reading) of mathematical statements, it might appear that the pursuit of these 'low-tech' mental games described in this paper is superfluous. Furthermore, as we said above, if HYPERPROOF manages to extend the traditional notions of syntax, semantics, logical consequence, and proof, notions that go far beyond the mere action of reading mathematical statements correctly, why would we want to re-invent the wheel? We do not. We are interested in pursuing the approach of these games precisely because they are 'low-tech' and limited in scope. They can be used anywhere, they can be applied to any setting.

A pedagogical suggestion to improve student understanding of multiple quantification arises out of our data relative to the game. It could be that, just as asking a student to express a statement as a computer program helps her or him make use of the syntax to analyze it, asking the student to play the game related to the statement leads to using the rules of the game as a syntax. Continuing our analogy, we can imagine that to the task of moving from the syntax of a programming language to the syntax of the statement and of mathematical discourse corresponds the task of moving from the rules of the game to the mathematical syntax.

A final pedagogical suggestion is related to the fact that we saw several examples in which a student showed a good understanding of the statement both within and without the context of the game. Our data does not permit us to decide in every case if one of these understandings contributed to the other. But even if it was the student's understanding of the statement that led to her or his understanding of the corresponding game, it would be possible to make pedagogical use out of the situation. It might be possible to use this "joint understanding" to transfer the student's understanding to other, similar but still troublesome, logical statements.

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A Questionnaire

Following is the questionnaire given to the students. It is an exact copy of the instrument that was used except that we omit here the space left for student responses.

Questionnaire

This questionnaire is to be filled out at one continuous sitting. There is no time limit. Please do not consult with anyone or any book (or other source) while you are still working on your answers.

Here are 11 statements. Decide for each one if it is True or if it is False, and indicate your answer by circling one of them. Then, underneath the statement, provide a brief explanation for your answer.

1. Everyone hates somebody. True False

   Why?

2. Every pot has a cover. True False

   Why?

3. Someone is kind and considerate to everyone. True False

   Why?

4. There is a mother for all children. True False

   Why?

5. All good things must come to an end. True False

   Why?

6. There is a magic key that unlocks everyone's heart. True False

   Why?

7. All medieval Greek poems described a war legend. True False

   Why?

8. There is a perfect gift for every child. True False

   Why?

9. There is a fertilizer for all plants. True False

   Why?

10. For every positive number a there is a positive number b such that $b < a$. True False

    Why?

11. There is a positive number b such that for every positive number a $b < a$. True False

Why?

How much time did you spend on this questionnaire? _______________

B Questionnaire Data

Number of students: 63.

TABLE 1

Statement	1	2	3	4	5	6	7	8	9	10	11
authors	AE	AE	EA	EA	AE	EA	AE	EA	EA	AE	EA
% same	84	92	65	16	83	22	76	27	29	59	19
% reverse	0	0	11	81	1	37	3	40	38	1	22
% unclear	16	8	24	3	16	41	21	33	33	40	57
% valid	89	92	81	95	76	68	76	84	74	41	9
% True	24	40	60	83	40	54	14	65	79	68	46

In the table above, **authors** refers to the authors' interpretation of each statement; **% same** refers to the percent of students who had the same interpretation of a statement as that of the authors; **% reverse** refers to the percent of students who had the opposite interpretation of a statement from that of the authors; **% unclear** refers to the percent of students whose statement interpretation was unclear; **% valid** refers to the percent of students who gave a valid argument (given their own interpretation for Statements 1—9, and given the authors' (mathematics) interpretation of Statements 10 and 11); **% True** refers to the percent of students who circled True.

TABLE 2 — AE versus EA

	as EA	as AE
% stdts interpreting at least 1 EA	92	94
% stdts interpreting all 6 EAs	0	0
% stdts interpreting at least 1 AE	5	97
% stdts interpreting all 5 EAs	0	35

TABLE 3 — AE versus EA and validity of arguments

	S	**SV**	**R**	**RV**	**U**	**UV**
EA stmts: 378	113	102	144	127	121	36
AE stmts: 315	248	235	4	1	63	5

The number 378 in the leftmost column above represents the total number of EA statements collectively among all 63 students ($= 63 \cdot 6$), and 315 is the total number of AE statements among all students ($= 63 \cdot 5$). The **S**-column lists the number of times students clearly interpreted the statements the same way as the authors and the **SV**-column lists the number of times these were given valid arguments. The **R**-column lists the number of times the student interpretation was the reverse of the authors' and the **RV**-column lists the number of times these were given valid arguments. Finally, the **U**-column lists the number of times the student interpretation was unclear, while the **UV**-column lists the number of times these were given a valid argument.

TABLE 4 — Valid arguments in natural language.

n	1	2	3	4	5	6	7	8	9
% $C(n)$	100	98	98	98	97	92	78	50	25

$C(n)$ is the set of students who gave at least n valid arguments for natural language statements.

Truth and Falsity

- Percent of statements that were interpreted as AEs and were declared True: 30%.
- Percent of statements that were interpreted as EAs and were declared True: 10%.

TABLE 5 — More on Statements 10 and 11.

% 10 right	41
% 11 right	9
% 10 & 11 right	8
% 10 right & 11 wrong	33
% 10 wrong & 11 right	2
% 10 & 11 wrong	57

Comments

1. Percent of students who claim 10 and 11 are the same or reworded: 42.
2. Percent of graduate who claim 10 and 11 are same or reworded: 33.
3. Percent of undergraduate students who claim 10 and 11 are same or reworded: 44.
4. Number of students who claim 11 is a rewording of 10 and give 10, 11 different truth values: 0.
5. Percent of students who explicitly state "they do not know" for Statement 7 but give a valid argument: 51.
6. Percent of students who explicitly state "they do not know" for Statement 7 but give a valid argument for 7 and also claim Statement 10 is the same as Statement 11: 18.

TABLE 6 — Mathematics vs. English

n	1	2	3	4	5	6	7	8	9
$R(n)$	92	92	92	92	92	93	92	94	94
$B(n)$	57	57	57	54	56	55	92	94	94

- $R(n)$ is the percent of students who got at least one of Statements 10 or 11 wrong or blank among those in $C(n)$.
- $B(n)$ is the percent of students who got both Statements 10 and 11 wrong or blank among those in $C(n)$.