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DEVELOPMENT OF THE PROCESS CONCEPTION OF FUNCTION

ABSTRACT. Our goal in this paper is to make two points. First, college students, even those who have taken a fair number of mathematics courses, do not have much of an understanding of the function concept; and second, an epistemological theory we have been developing points to an instructional treatment, using computers, that results in substantial improvements for many students. They seem to develop a process conception of function and are able to use it to do mathematics. After an introductory section we outline, in Section 2, our theoretical epistemology in general and indicate how it applies to the function concept in particular. In Sections 3, 4, and 5 we provide specific details on this study and describe the development of the function concept that appeared to take place in the students that we are considering. In Section 6 we interpret the results and draw some conclusions.

1. INTRODUCTION

We take it that "understanding the function concept", if it is to go beyond mere manipulation of formulas or playing with Venn diagrams, must include a process conception. We would like students to have the ability to make sense out of a situation by constructing a mental process that transforms (mental) objects. We are not the first to observe that a process conception does not arise spontaneously in human beings in our culture, nor does a large number of years of schooling have much effect in this regard. See, for example, Sfard (1987) and Even (1988). The data we present in Section 3 is only one more confirmation of such observations.

Our intention in this paper is to go beyond the observation and classification of student difficulties with functions. We try to interpret these difficulties in terms of a theoretical perspective and use this interpretation to design an instructional treatment. According to our theoretical viewpoint, a major requirement for understanding functions that students do not seem to meet is the ability to construct processes in their minds and use them to think about functions. We present data suggesting that working with computers in certain ways seems to have, in general, a salutory effect on students' abilities to make such constructions. We show that they were not able to make them before our instructional treatment. Moreover, and we suggest it is a consequence of their new abilities, after the treatment they are successful in doing mathematics that is considerably more difficult than students of their level are capable of handling. The instructional treatment

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that we use is derived from our theory, and this relationship is described in Section 5.1. In Section 5.3 we apply our theoretical perspective to organize and interpret the vast amount of data obtained from our observations. We try to show some examples of how students appear to make constructions called for by the theoretical analysis, and we suggest this is because the instructional treatment focused on those very constructions. We feel the outcomes support our theoretical perspective.

The students we are concerned with in this paper were, for the most part, pre-service teachers – largely secondary, but some elementary majors. We are interested in somewhat more sophisticated understandings of functions than are usually required or expected of such students. These include using functions to analyze reasonably complex situations, performing operations with specific functions given in complicated ways (e.g., composing or multiplying two functions with different split domains), theoretical issues such as preservation of properties like *1-1* or onto when functions are combined in various ways, and functions whose domains and ranges are sets of functions. Finally we point out that this is not an isolated, laboratory experiment. Our instructional treatment and all of our observations took place in the context of two instances of a one-semester course on discrete mathematics. There is a textbook, Baxter et al. (1988), which supports our approach and this course has been given on several occasions by various instructors at various institutions.

There is a vast literature on learning the function concept and we refer to Leinhardt et al. (1990) for a survey. Our work has benefited from studies of student difficulties such as in Thomas (1969), Orton (1970), Herscovics (1982) and Vinner and Dreyfus (1989); taxonomies of the function concept as in Lovell (1971) and Dreyfus and Eisenberg (1983); considerations of curriculum sequencing such as Buck (1970) and Freudenthal (1982); theoretical analyses as in Sfard (1987); microworlds as developed by Schoenfeld et al. (1990); and studies of multiple representations as in Kaput (1987).

We rely very heavily on Piaget's fundamental study of the function concept (Piaget et al., 1977) and his overall theory (see for example, Piaget, 1975). He studied subjects ranging from ages 4 to 14 and was interested in epistemological relations among the concepts of functions, causality and what he calls operations. He was, in Piaget et al. (1977), mainly (but not exclusively) concerned with very simple functions of the form $f(x) = ax$ and tried to show how such functions could lead to the notion of proportion. Our work differs from Piaget in that we are concerned with older students and the design, implementation, and evaluation of instructional treatments based on theoretical investigations.

Our work in this area began with Ayres et al. (1988) and is closely related to Dubinsky and Lewin (1986) and Dubinsky et al. (1989). The students in the present study were mainly sophomore and junior math majors preparing to be high school, middle school or elementary school math teachers. Before beginning the course, they had taken a fair amount of undergraduate mathematics including the full calculus sequence. Their performance in mathematics could, perhaps, be called mediocre. There were 62 students in this program.

2. EPISTEMOLOGY OF FUNCTIONS

The mathematical epistemology we are developing in conjunction with studies such as the one reported in this paper is described in a number of papers such as Dubinsky (1991). We give here only a brief mention of the general theory and how we attempt to apply it to the concept of function. We begin with the following statement (Dubinsky, 1989).

> A person's mathematical knowledge is her or his tendency to respond to certain kinds of perceived problem situations by constructing, reconstructing and organizing mental processes and objects to use in dealing with the situations.

There is not room for a discussion of the several important issues raised by this statement. Our particular interest in this paper is that applying this point of view to mathematics (or any other subject) consists of determining the nature of the specific processes and objects that are constructed and how they are organized when one studies mathematics. It is the nature of these objects and processes that gives the theory a mathematical flavor – if that is the area to which it is being applied.

We first provide a general discussion of mathematical objects and processes in our theory. This will then be related to functions. Our point of view is illustrated in Fig. 1 where we show the various ways in which objects and processes are constructed. These means of construction are called, in this theory, *reflective abstractions*. They are discussed in some detail in Dubinsky (1991a,b).

An *action* is any repeatable physical or mental manipulation that transforms objects (e.g., numbers, geometric figures, sets) to obtain objects. When the total action can take place entirely in the mind of the subject, or just be imagined as taking place, without necessarily running through all of the specific steps, we say that the action has been *interiorized* to become a *process*. It is then possible for the subject to use the process to obtain new processes, for example by reversing it or coordinating it with other processes. This is not a completely satisfactory description of the distinction

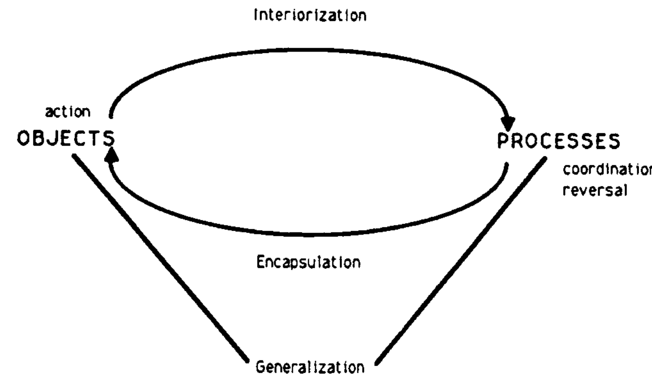


Fig. 1. Construction of objects and processes.

between an action and a process. We consider such a specification to be an important open question which will require additional study, and it will be the object of future papers. We give here the understanding of this distinction with which we began our study and, in Section 6.4, we describe an improved, more operational version based on the work described in this paper. Finally, when it becomes possible for a process to be transformed by some action, then we say that it has been *encapsulated* to become an *object*.

According to this theory, although there are several ways to construct processes (interiorize actions, reverse or combine processes) there is only one way to make a mathematical object – by encapsulating a process. The importance of this lies in the fact that in many mathematical situations it is essential to be able to go from an object back to a process. One of the tenets of the theory is that this can only be done by *de-encapsulating* the object, that is, to *go back to the process which was encapsulated in order to construct the object in the first place.* (Note that we are not asserting that encapsulation and its reversal really is the only way to go from a process to an object and back. We are only saying that this is the only way that our theory provides for. The larger question is then part of evaluating the theory.) The presence of *generalization* in Fig. 1 is just an indication of the fact that in some cases, even when the situation is new, little or no additional construction is necessary. Existing objects and processes can be used to deal with the new situation. The only learning that takes place here is that the tools that one already has can be used to handle a new situation.

We now relate our general theory to the specific mathematical concept of functions. In this paper, our emphasis will be on function as process. There will be only a brief mention of the object conception as it relates to processes. We consider three ways of thinking about functions: prefunction, action, and process.

For *prefunction* we consider that the subject really does not display very much of a function concept. Whatever the term means to such a subject, this meaning is not very useful in performing the tasks that are called for in mathematical activities related to functions. We will present some examples of prefunction conceptions in the next section.

An *action* is a repeatable mental or physical manipulation of objects. Such a conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression). A student whose function conception is limited to actions might be able to form the composition of two functions given via algebraic expressions by replacing each occurrence of the variable in one expression by the other expression and then simplifying, but he or she would probably be unable to compose two functions in more general situations, e.g., when functions had split domains, or if they were not given by expressions at all.

A *process* conception of function involves a dynamic transformation of objects according to some repeatable means that, given the same original object, will always produce the same transformed object. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. When the object has a process conception, he or she will be able, for example, to combine it with other processes, or even reverse it. Notions such as *1-1* or onto become more accessible as the subject's process conception strengthens.

We note that the transformation in a process can be completely mental and so we are not restricting ourselves to computable functions in the mathematical sense. In terms of a single individual whose understanding of functions is being constructed, one may consider that an action conception of function is a sort of "pre-process" conception. This means, of course, that many individuals will be in transition from action to process and, as with all cognitive transitions, the progress is never in a single direction. This makes it quite difficult, in any but extreme cases, to determine with certainty that a particular individual's function concept is limited to action or that he or she has a process conception. As a result, any interpretations of individual responses must be in broad terms and fine distinctions cannot be made with much reliability.

3. RESULTS OF ORDINARY SCHOOL ENVIRONMENTS

We attempted to observe the presence of a process conception in several ways. One way was to ask the students to respond in writing to the question, What is a function? This took place in class at or near the very beginning of a semester. In a second question, which was given immediately after they responded to the first question, the students were asked to give examples of a function.

WHAT IS A FUNCTION?

The responses to the first question were grouped into four categories: prefunction, action, process and unknown. The categories used here are repeated in the second observation (Section 4.3). A prefunction response is one in which it appears that the student does not have very much of a function concept at all. We assigned a response to this category for example, if it was to the effect that "I don't know", or "a mathematical equation with variables", or "a mathematical statement that describes something" or a "social gathering". We categorized as actions responses that emphasized the act of substituting numbers for variables and calculating to get a number, but did not refer to any overall process of beginning with a value (numerical or otherwise) and doing something that resulted in a value. Any response in which explicit mention of beginning or of resulting objects was missing was categorized as an action conception. If all three were present, but the procedure was tied to an expression or equation, or if the input or output objects were strongly restricted, say to integers, then the action category was chosen. Here are some examples of the kinds of responses that were put in the action category.

A function is something that evaluates an expression in terms of x.
A function is an equation in which a variable is manipulated so that an answer is calculated using numbers in place of that variable.
A function is a combination of operations used to derive an answer.
A function is an expression that will evaluate something when either variables or numbers are plugged into the function.

In process responses the input, transformation, and output were present, integrated and fairly general. Here are some examples.

A function is a statement that when given values will operate with these values and return some result.
A function is some sort of input being processed, a way to give some sort of output.
A function is an algorithm that maps an input into a designated output.
A function is an operation that accepts a given value and returns a corresponding value.

In addition to these categories, there were some responses in which we were unable to tell if a process conception was present. These responses are listed as unknown. Here is a tabulation of the four categories of responses. The results are given as percentages of responses listed in a given category.

TABLE I
Responses to "What is a function" – before instruction

Prefunction	Action	Process	Unknown	Total number of students
40%	24	14	21	62

It is important to mention that a number of responses suggested a combination of categories. What this indicates is that our four categories do not represent "stages in development" of the function concept, but rather, different ways of thinking about functions. They may not be mutually inconsistent and can coexist. A particular conception can also exist in different forms, some of which are of greater mathematical sophistication and some less so. In this paper, we deal with the combinations of conceptions as follows. Prefunction is clear in the sense that, by definition, no response which suggested any additional conception would be considered prefunction. If a response indicated action or process but not both, then it was so categorized no matter what else was present. If both were indicated, then it was placed in the process category. The remaining responses were categorized as "unknown".

EXAMPLES OF FUNCTIONS

The students were asked to give examples of functions, and in some cases they were specifically asked to try to give examples of different kinds of functions. The total of 106 examples fell into the following 8 categories:

I. Something that could not really be interpreted as a function.
II. Omitted.
III. An equation in two or more variables.
IV. A graph.
V. $F(x) =$ some algebraic or trigonometric expression.
VI. $y =$ some algebraic or trigonometric expression.
VII. An algebraic or trigonometric expression alone.
VIII. Some attempt to describe a process.

In Table IIa, the responses are again tabulated as percentages.

TABLE IIa
Examples of Functions – before instruction

I	II	III	IV	V	VI	VII	VIII	Number of examples
8.6	12.4	0	0	41	23	3.7	7.5	106

It is possible to consider V, VI and VII as a single category. Although one might think that using the terminology $F(x)$ indicates a process conception, we concluded, after comparing this response with other responses made by the same students, that it is little more than a remembered notation, and we consider that this grouping is more suggestive of an action conception. It is also possible to merge the first three categories. Ignoring Category IV (which hardly occurred), one might consider that essentially all responses fell into only 3 categories: prefunction, expression or action, and process. With this grouping, the percentages are tabulated as follows.

TABLE IIb
Examples of Functions – before instruction

Prefunction	Action	Process	Number of examples
26	67	3.7	106

The examples given appear to correspond somewhat with the students' statements of what a function is, the main difference being a movement from prefunction and unknown to action. There is also a small increase in the percentage of responses that indicated a process conception. It is neither surprising nor unusual that the students' choice of examples indicates what could be a slightly more sophisticated conception of function than did their explanation of what a function is. There is a lot of literature describing and explaining the fact that one can often do something successfully without necessarily understanding (which is more than merely describing) how it is done (Piaget, 1976, 1978). On the other hand, we will see below a case in which the reverse occurs (Tables III and IVb). We will offer interpretations of this anomaly in Section 6.1. In any case, the data here are consistent with the weak conception of function that we saw in the previous section.

4. RESULTS OF A GENERAL COMPUTER ENVIRONMENT

After a large portion of the course in elementary discrete mathematics which is one of the main concerns of this project, students were subjected to the same two observations as described in the previous section. That is, they were again asked to respond to the question, "What is a function" and to give three different kinds of examples of functions. The observations were taken just before beginning a unit on functions. This was about two-thirds of the way through the semester.

One of the goals of the first portion of the course was to create an environment conducive to students' spontaneously constructing mathematical concepts. This was done through a combination of computer experiences and small group problem solving. The instructional treatment in the first portion of the course was not focused on functions. It was about learning the syntax of **ISETL**, working with elementary properties of number systems, propositional calculus and sets. Functions arise naturally when studying such mathematical ideas and **ISETL** lends itself to using functions explicitly in working with them. This was exploited and, occasionally, the terminology of functions was introduced informally as a way of talking about one or another phenomenon. Thus, we had an environment which contained functions as a natural part of its background, but did not consider them explicitly. In the rest of this section, we describe the general computer environment in a little more detail, and then look at what happened to the students' conceptions of functions as a result of working with computers for about two months.

4.1. *The Environment*

ISETL is an interactive, interpreted programming language that implements a number of mathematical constructs in a syntax which is very similar to standard mathematical notation. It runs on Macintosh and MS-DOS computers. One uses **ISETL** by entering an expression to which the system responds by evaluating and returning a result. An expression can involve arithmetic operations on numbers (integers or decimals), boolean operations, or operations on character strings. Assignments can be made to variables and expressions can combine variables and constants. The domain of a variable is determined in context dynamically (it can change) by the system and there is no need to declare data types, sizes, etc. Many important mathematical operations on these data types are implemented directly in **ISETL** and are used with a single command. In addition to arithmetic, they include mod, max/min, even/odd, signum, absolute value, random, greatest integer less than, concatenation (of strings) and the standard trigonometric, exponential and logarithmic functions. **ISETL**'s power appears with the complex data types of set, tuple, string, func, and smap. Syntax such as

```
{7 .. 23};
{-4, -1 .. 40};
{9, 7 .. 0};
```


can be used to construct sets of finite arithmetic progressions of integers. It is also possible to construct a set containing any data types whatsoever (including other sets) simply by listing them. For example, the following set has cardinality 5:

```
{8 - 1, "t" + "he", 1.2, {1, 3, 4, 2}, {1 .. 4}, 3 < 2, "the", 7, false};
```

Once such sets have been constructed, one can then construct complicated subsets by using a set former notation that can generally be understood by anyone who knows the mathematics. For example, here is the set of cubes of the integers of absolute value less than or equal to N whose squares are congruent to 2 mod 4.

```
{k**3 : k in {-N .. N} | k**2 mod 4 = 2};
```

Standard set operations are implemented with single commands. They include union, intersection, difference, adjunction, tests for membership or subset, power set, cardinality, and selection of an arbitrary element. It is possible to iterate over a set to make loops but the operations of existential and universal quantification over a set are implemented and they often render loops superfluous. For example, the following two lines construct the set of all primes less than or equal to N and then the last line verifies the Goldbach conjecture up to N. The last line returns the value true.

```
E := {4, 6 .. N};
P := {p : p in {2 .. N} | (forall q in {2 .. p - 1} | p mod q /= 0};
forall n in E | (exists p, q in P | n = p + q);
```

The **ISETL** tuple is a finite sequence. It may be considered as an infinite sequence, only finitely many of whose terms have been defined. One can change a single component of a tuple, concatenate two tuples, or add a term to the end. If any of these operations affect the length of the tuple, the change is made automatically. The same iteration and quantification operations can be performed over the tuples as well as sets. The standard notation for access or evaluation is used for tuples. Thus, if t is a variable which has been assigned a tuple as its value, then t(3) is the value of the third component of t. The syntax for tuples is very similar to that for sets except that square brackets [ ] are used instead of curly brackets. Moreover with sets, there is no notion of access or evaluation "at a point". The string is similar to the tuple, but the components must be characters.

The name for procedure in **ISETL** is func. A func represents a function; it accepts parameters and returns a value. Here is an example of a func which represents multiplication by 5 mod 6

```
m5_6:=func(x);
        if x in {0 . . 5} then return 5*x mod 6; end;
      end;
```

A func may be assigned to a variable (such as m5_6 in the above example) and then standard evaluations such as m5_6(4) have the usual meaning. As we indicated above, students work with such material and the language of functions will be used, but it arises in the context, here for example, of learning about modular arithmetic. It is also possible to implement a function in **ISETL** as an smap which is a set of tuples each of which has length 2. For example, the above function can be represented with an smap,

```
m5_6s:={[0, 0], [1, 5], [2, 4], [3, 3], [4, 2], [5, 1]};
```

or

```
m5_6s:={[x, 5*x mod 6] : x in {0 . . 5}};
```

Again, the standard evaluation notation can then be used.

The syntax for evaluating a tuple, string, func or smap at a point is, as in mathematics, all the same. The operation of evaluating funcs which the student has constructed is the main tool underlying all of the computer work aimed at helping students develop a process conception of function. Both funcs and smaps are data and can be treated as such. They can be passed to procedures as parameters, constructed by a procedure and returned, or operated on as with ordinary data.

The students spent this two month period studying various topics in mathematics in the context of working with **ISETL**. They worked with modular arithmetic, greatest common divisor, change of base, use of random numbers, decimal and fractional representation of rational numbers, representation of logical statements (mainly first order – very little quantification) and determination of truth values, validity arguments, various means of proof, manipulation of sets, and expression of complex mathematical statements (e.g., Goldbach's conjecture) by means of set theoretic expressions. A combination of two teaching methodologies was used. The students were given various tasks to perform on the computer as well as with paper and pencil. These were to be done in lab and as homework. The tasks were aimed at raising questions in the students' minds and confronting them with situations for which it was considered that the construction of a particular mathematical concept, or a new aspect

of it would be a good way to make sense out of it. In class, the students were divided up into small groups of 4 or 5 and set to work on similar tasks as a group. After some time, there would be general discussion of the task by the whole class, led by the instructor. Rarely was a full explanation offered by the instructor and, on occasion, questions that had been raised were left to the students' judgements and choices between possible answers were often not made explicitly. The goal was to create an atmosphere of openness, uncertainty, and cooperation. The students were urged to reflect on what they were doing and to think of mathematics more as the development of ideas rather than the search for specific answers.

4.2. *Instruments*

Three instruments were used to investigate the students' process conception of function after the period of general computer experience. The first was to put to them again the question of what is a function. The second was given at the same time and was a request for three different kinds of examples of functions. For the third, given some time later, they were given a list of descriptions of situations. (See Appendix I.) Their task was to decide if the situation could be expressed by functions and if so, to explain how.

4.3. *Results*

WHAT IS A FUNCTION

Here is a tabulation of the responses of the students to the question "What is a function" after the general computer experience but before the unit on function. The categorization of responses was done exactly as in Section 3.

TABLE III

Responses to "What is a function" – after general computer environment

Prefunction	Action	Process	Unknown	Number of students
17%	36	36	11	59

The most striking feature that we can observe in comparing these responses with the responses of the same students before their experience in a general computer environment (Table I), is the very sharp drop in prefunction responses. These are distributed among the richer action and process responses. There is also a drop in the number of responses classified as unknown.

EXAMPLES OF FUNCTIONS

Examples given by the students after their two months in a general computer environment suggest that these activities may have enriched their overall mathematical experience and made an impression on their thinking about functions. Of the 170 examples given by 59 students, 69 involved **ISETL** operations, either those that are predefined in the language (mod, abs, etc.) or processes which the students had constructed in the course of their computer work. Thus altogether, 69 out of 170 examples, which is more than 40%, related to **ISETL** operations. In order to express the increased richness of examples in our tables, we break down the categories used in Table IIa a little further. The Roman numerals remain the same, but numbers V, VI, and VII are decomposed into sub-categories. Here is the new categorization scheme.

I. Something that could not really be interpreted as a function.
II. Omitted.
III. An equation in two or more variables.
IV. A graph.
Va. $F(x)$ = some algebraic or trigonometric expression.
Vb. $F(x)$ = some **ISETL** expression.
VIa. y = some algebraic or trigonometric expression.
VIb. y = some **ISETL** expression.
VIIa. An algebraic or trigonometric expression alone.
VIIb. An **ISETL** expression alone.
VIIc. The name of an **ISETL** representation of a function.
VIII. Some attempt to describe a process.

TABLE IVa

Examples of Functions – after general computer environment

I	II	III	IV	Va	Vb	VIa	VIb	VIIa	VIIb	VIIc	VIII	No. of examples
11%		9	1.6	26.9	4.1	7.8	2.2	4	8.6	8.3	11.7	170

We can use exactly the same merging as in Table IIb.

TABLE IVb

Examples of Functions – after general computer environment

Prefunction	Action	Process	Number of examples
24.8%	62.2	11.7	170

We do not see as many clear indications in these two tables as we did in Tables IIa,b. There are some observations that seem consistent with our theory and what we have been seeing, but some of the data may require other explanations. If we compare Table IIb with Table III, we see a reverse of what was contained in Tables I and IIb. Here, the first column, which indicates a lack of any reasonable function concept, is higher for examples than for explanation. Also, the ratio of action to process responses is higher for examples than for explanations. One could say that the students expressed stronger process conceptions of function in providing explanations than they did in giving examples. This is the opposite of what we found in Tables I and IIb before the general computer environment. We will offer interpretations of this in Section 6.1.

FUNCTIONS IN SITUATIONS

Tabulation of scores. The 24 situations are organized in 9 categories listed in the first column of Table V, corresponding to the nature of the situation. The numbers in parentheses locate the situation in the list given in Appendix I. Table V gives a tabulation, by category and exemplar within the category, of the percentages of students who could see functions in situations. This is a very rough measure relying basically on a yes/no response with minimal attention given to the nature of the function or why the student thought there was or was not a function.

Even with this superficial observation there are some trends that seem to emerge. The most striking feature is the very low scores. Overall, the students said yes about 40% of the time. It should have been closer to 100%. The results vary sharply with the category. Not surprisingly, the parameter equations and the funcs are among the most likely to be seen as representing functions. In these situations it is hard to miss seeing a process

TABLE V

Functions in Situations

	Exemplar	Exemplar 2	Exemplar 3	Exemplar 4
Funcs (7, 22)	76.5%	74.6		
Tuples (2, 10, 23)	55.2	54.1	61.6	
Smaps (1, 8, 14, 17)	50	38.4	27.4	37.5
Strings (5, 21)	4.1	2.8		
Parametric (18)	57.3			
Equations (4, 9, 13, 15)	25	29.5	25.7	31
Graphs (3, 6, 16, 24)	40.7	24.6	23.3	19
Tables (12, 20)	39.9	48.6		
Physical (11, 19)	30.2	35.8		

so the only thing that is really indicated is that the notation (**ISETL** for funcs and mathematics for parametric equations) was understood. The response for tuples also indicates that a large majority of the students may have learned that this **ISETL** construct represents a sequence and this is one kind of function. Again, this is fairly straightforward. You have to think a little more to see a function in an smap, equation, graph, table or physical situation. Less than one-third of the students did. This seems to have little to do with **ISETL** or even this course. Graphs and equations are standard topics in the undergraduate curricula that these students studied, yet no more than about 1/4 of them saw functions in such situations. They did only a little better with smaps and tables. Essentially none of the students saw a function in the strings.

Seeing a process when it is not explicit. Although the explanations that the students gave were brief, even a cursory reading of their comments suggests that they understood both the **ISETL** syntax and the mathematical notation, but their process conception had not developed very much. As a group, they did not show the ability to construct a process in their minds in response to a situation. If, however, the process was explicitly described in the situation, as is the case with funcs and the parametric equations, they did seem to understand this and were willing to say that there was a function. In all of the other categories, no process is explicitly described and so the subject must do a great deal to construct one that relates to the situation. They did not seem to do this. Even when students did say there was a function, their comments did not suggest very often that they were building a process in their minds. The responses *do* suggest that they *were* looking for processes. But in general their thinking was global and tended to be external to the processes that mathematicians construct in their minds in order to analyze situations. We can identifiy some specific obstacles to constructing processes. Many students insisted that there be an expression or at least the presence of variables to indicate "input" and "output". In many cases they insisted on the presence of causality before they were willing to construct a process. We will illustrate some of their reactions in terms of the individual categories.

For the tuples and smaps, many students seemed to be only responding to the presence of an expression when they said there was a function. For the process, they simply referred to the overall construction of the object. They rarely referred to the process of going from the index to the component in a tuple or from the first component of a pair to the second in an smap. Often, the construction they were concerned with was only the

set formation. For the equations in two variables, most students simply tried to solve for one variable in terms of the other. If successful, they said it was a function. Otherwise, they said it was not. Some students used an interesting function process for equations which they, themselves, had brought up in class. They spoke of putting in (one or two) values for the variables, testing the equation and returning true or false as the value of the function. It occurred with about 15% of the students. For both the graphs and the tables, many students tried to guess a formula that described the relationship before they were willing to agree that there was a function. Sometimes they seemed to construct a process very definitely, but would then reject it if they could not find a formula or at least imagine some causal relationship. In the physical situations and the strings, the students concentrated on the global meaning of the situation and did not generally make any analysis of the components to construct functions.

Summary. In all, these results suggest that after weeks in a general computer environment, these students became familiar with using the syntax of **ISETL** to represent functions and seemed to realize that the thing to do was to look for a process. They were not, however, very good at finding or constructing processes on their own if there was not an explicit algorithm or expression. They looked for processes in equations and causality, not in mental transformations of mental objects.

5. RESULTS OF AN INSTRUCTIONAL TREATMENT FOR FUNCTIONS

5.1. *The Treatment*

The students were subjected to an extensive instructional treatment of functions over a four week period. Within the context of the teaching methodologies described above (computer experiences and small group problem solving) an attempt was made to explicitly foster the specific mental constructions that our theory calls for. In the case of the process conception of function, this meant that we had the following goals for the students:

1. Respond to a function situation by constructing a mental process that involves input objects and a transformation of these objects into new objects.
2. Expect complete generality both for the objects and the processes. In the case of objects, this would include things that were new in this role for the

students such as boolean values, sets, as well as numbers of all kinds. For processes, this was to include any means, whether it be something that could be imagined, an explicit recipe such as an **ISETL** construct, or a formula on paper, for going from given objects to new ones.

3. Connect the internal mental construction with external representations and connect the external representations with each other. The external representations considered in the course included: mathematical description which is a specification of domain, rule of transformation and range; graph; **ISETL** func; smap; tuple; and string.

4. Be consciously aware of the function process as a total mental activity. Think about this process both in terms of individual examples and as a general construct. Work with the process in a forward direction. Understand various specific mathematical topics relating to functions in terms of the function process.

5. Coordinate two or more function processes. This includes, but can go beyond, the composition of functions to more complicated combinations as in finite state automata.

6. Reverse the function process. Think about the reversed function process both in terms of individual examples and as a general construct. Understand various specific mathematical topics relating to functions in terms of a reversed function process.

We will describe the activities that were designed to meet each of these goals. We will refer mainly to problems, computer activities and class discussions. The problems occurred both in class (to be worked on in small groups) and as homework. The computer activities occurred in a lab situation (again to be worked on in small groups and in the presence of assistants who would answer questions – usually by asking appropriate questions) and as homework. They were:

1. There was a class discussion of each of the examples in the Functions In Situations instrument and how it could be considered to be a function. The students were given a number of problems in which they were asked to look at an **ISETL** representation of a particular function and give a mathematical description. Conversely, they had problems in which they were expected to construct the **ISETL** representation from the mathematical description. The idea behind this activity was that the act of constructing a computer implementation of an action would tend to lead the student to interiorize a mental process in connection with this action. Students were urged to think consciously about what they were doing.

2. Most of the work of getting students to think of functions in a context that went beyond substituting numbers in explicit algebraic or trigonometric expressions to get answers took place in the period before the study of functions per se began. At no time were students told that the objects that resulted from applying a function's process could be boolean values (i.e., true or false), or that the process itself did not need to be monolithic but could do different kinds of things depending on what kind of input was given. The students worked with situations of this kind on the computer in the two months before the study of functions began and, judging from their examples in the second observation, one might conclude that they had made a number of mental constructions which they had incorporated into their thinking about inputs, outputs, and processes.

3. Students already began constructing connections between various representations when they were working on constructing mental processes as a way of understanding something that, for the teacher, is an external representation of a function. Activities designed to help students make these connections explicitly had to do with writing an **ISETL** func that would accept a representation of one kind and return a representation of another kind.

4. Awareness of the function process came through working problems that required students to think hard about it. Here is one involving evaluation of functions. It is difficult and we believe that struggling with such difficulties helps develop a process conception of function. The students are given the following tuple (sequence) of tuples of strings

```
t:= [["Tom", "Dick", "Harry"], ["Peter", "Paul", "Mary"],
     ["Mutt", "Jeff"]]
```

and asked to find `t(2..3)(2..)(1)(2..)(1)(2)` (Here, `t(2..3)` refers to the tuple of the components of `t` from the second to the third, and evaluating a tuple `s` over `2..` produces the tuple of those components of `s` from the second to the last.) Not everyone gets the answer "e" the first time.

5. The students are given a thorough workout in thinking about the forward process by working on problems and applications (e.g., relational data bases and finite state automata) connected with equality of functions (they are asked to write funcs that will test for it), images (they are asked to compute them for functions given by various kinds of representations), pointwise arithmetic of funtions (they are asked to write funcs that accept representations of two functions and return a representation of their pointwise sum, difference or product), and composition (in addition to a func that will compute it, they are asked to compute mentally a number of compositions of functions given by fairly complicated descriptions).

6. There are activities designed to help the students think about situations in terms of reversing the process of a function. These are embodied in problems that compute pre-images of functions, the construction of an inverse function and the concepts of *1-1* and onto.

5.2. *Instruments*

After the instructional treatment was completed, three separate instruments were used to obtain a picture of the students' new conceptions of functions and how they had changed.

SOME QUESTIONS ABOUT FUNCTIONS

The first of these "post-treatment" instruments is a collection of questions about functions. This was given in an "exam-like" context in which students were required to work individually without the benefit of notes, texts, or computers. The students came to school one evening and were given unlimited time to answer the questions. They did this voluntarily and were told that the results would not have any negative effect on their grades. The observation took place almost immediately after the instructional treatment was completed. The items are given in Appendix II. Some of the questions were designed to determine the students' technical knowledge regarding **ISETL**. They are not relevant to the study, except to show that the students did not have serious difficulty with **ISETL** syntax, so we omit them.

The first question (Item 1) was designed to indicate the student's ability to construct a function and use it as a tool to gather and organize information about a given situation. There is no formula given nor might one expect to find any algebraic relationship although a vague kind of dependence is hinted at in the story that is told. One process that a student might construct is to respond to the "input" of a vowel by running through the sentence, counting the number of occurrences of that vowel. The result of this count is then the "output" of the function.

Item 2 is considerably more difficult. It requires the construction of three processes: the process determined by int_str which is actually described in the problem, the process of the func F which is given by the **ISETL** code, and the process of a function given by a set of ordered pairs. This latter is constructed inside the func and not explicitly provided for the student. After constructing these processes, the student must coordinate (compose) them to obtain a single process to make the evaluation in part (a). Parts (b), (c), and (d) require the student to coordinate two of these processes with the reversal (inverse) of the third. To answer

these four questions, the student might make these constructions and follow them through to get the answers. This is fairly complicated. Alternatively, he or she might interiorize the composed process and understand it as a whole such as: $F(a)(b)(c)$ is the cth character in the string which is the name of the integer given by the ath power of the integer b. Once this is accomplished, the problem is not very difficult. This is an example of a student using a powerful mental construction to make a complicated problem easier.

INTERVIEWS ABOUT FUNCTIONS IN SITUATIONS

The important thing about the Functions in Situations instrument is that most of the examples do not present an explicit process for a function. In every case, however, it is possible (and in all but a few both reasonable and mathematically useful) to construct a process that leads to a function that describes the situation. The question is about how well the students were able to do that.

After the instructional treatment was completed, a selection of 19 students were interviewed about some of their responses to the instrument made just before the instructional treatment began (see above). We attempted to see if the students had been able to construct processes in their initial answers, before the treatment, and how well they could do it at the time of the interview. While the interviews, which were conducted by one or another of the authors, did not have many set questions, the overall structure of each was similar. Each student was asked for her or his own definition of function. Then the main line of inquiry was to ask the student to explain what he or she could remember about the thinking that was behind the written response. We then asked how he or she would respond to selected situations now, after instructions – always trying to get the student to relate the response to the definition which he or she had given. The student was encouraged to change previous answers and to modify her or his definition of function. Interviews were recorded and transcribed. Copies are available from the authors on request.

FINAL EXAM

About 75% of the questions on the Final Exam for the course are appropriate for this study. They are listed in Appendix III. In general the Final Exam questions were designed to be more difficult than the items on the Some Questions About Functions instrument. On the Final Exam we tried to move even further away from what was discussed in class and towards more difficult applications of these concepts in mathematics. Indeed, all of the questions on the final exam are stated completely in terms of mathematics with no reference to computers. This was never done so strongly during the course.

Question 2 is concerned with students' ability to work with a process when it is given by a set of ordered pairs listed in a table. The particular task is composition. Question 2a only requires them to combine two given processes. Questions 2b,c require that they be able to reverse at least one of the processes before combining it with the other. All of the work is local in that it is concerned with applying the processes to individual points. There is an interesting point with respect to Question 2c. The answer in fact is no. This is because unless f is *1-1*, the number -1 might not be the only value which f sends to -2. On the other hand, it is possible to interpret this question as stating that the domains of f, h are both the set $\{-1, 2, 4\}$. If this restriction is made, then the answer to the questions is -1.

Question 3 is a straightforward calculation, but it can be difficult because the domains of the functions split in different ways. Understanding pointwise multiplication of functions is not enough. You have to understand the process in order to determine which branch to choose since it is different for the two functions. The question is designed to determine if the students' process conception is stable or if it will break down in face of such a difficulty.

Question 4 is a straightforward issue in theoretical mathematics. The students spend very little time (less than one class day) on the concepts of *1-1* and *onto*. They saw nothing like these two problems during the course. The question is much harder than one would ordinarily put on an examination for students at this level. If present at all, such questions would mainly be designed to identify the "A" students.

Question 5 moves into the domain of function as object. Parts (b) and (c) require the student to coordinate a process conception with an object conception and also to reverse the process. Thus, the student must think of the functions in $\mathscr{F}$ as objects so that D and K can act on them. In the expression $K(f)$, f is an object. But to understand the meaning of $K(f)(3)$ one must *imagine* the process which transforms x to $f(x)$ – without knowing anything about it – and construct the new process which transforms 3 to $f(-6)$. Again this is not a question one would expect undergraduates with mediocre ability in mathematics to be able to handle. It does relate to an important point in our epistemology of functions. It is necessary not only to encapsulate a process to obtain an object, but also to be able to unpack or *de-encapsulate* the object and return to the process – even to go back and forth at will.

5.3. *Results*

In this section we summarize the results of the instruments described in the previous section. Possible interpretations of these results will be considered below in Section 6.

SOME QUESTIONS ABOUT FUNCTIONS

The responses on this instrument were scored on the basis of 1 point for a correct answer and 0 otherwise. The percentage of correct answers for 56 students on each item is given in Table VI. The most striking thing about these results is the very high scores. None of the items appear to be exceptionally problematic for the class as a whole, the lowest score on any item being 65.7%.

TABLE VI
Some Questions About Functions – correct responses

Item 1	85.8%
Item 2a	73.2
Item 2b	65.7
Item 2c	75.3
Item 2d	69.3

Looking at individual questions, we note that nearly everybody (48 out of 56 students) succeeded in constructing a function and using its process to organize information about the frequency distribution in a text (Item 1). This is in sharp constrast to the difficulty most of them had before instruction in even thinking that a function was present, much less actually constructing it. We see very little trace of their earlier need for an explicit process, a formula or a clear dependency.

Looking at the remaining items, it seems that only a few students – presumably the weakest (recall that the mathematical ability of these students is, at best, mediocre) were unable to construct a very complicated combination of processes and then reverse them selectively. Incidentally, recall that Item 2c has two correct answers, 1 and 10. A total of 7 of the 56 students pointed this out. On Item 2d, a total of 31 of the 56 pointed out that any positive integer will work. Of course they were only asked for a single answer, so it is possible that more students were aware of the duplications but decided not to mention them.

INTERVIEWS

The interviews, which took place after instruction, provide a rich source of illustrations of process conceptions of function on the part of many students. In some cases, we see this in students who had very weak process conceptions before instruction. We can even see examples in which students found it necessary to (re-)construct their function concept in response to confusion they experienced during the interview. Often they are conscious of having dismantled previous (inadequate) understandings, replacing them with more powerful conceptions. This is particularly striking in the case of the familiar difficulty of confusing single-valued with *1-1*. In the following pages we analyze portions of some of the interviews in order to relate the results of instruction to our theoretical perspective.

Expressing an action conception. The following excerpt is an example of what we mean by an action conception of function. In looking for a repeatable mental or physical manipulation of objects for Appendix I, Situation 13, the student indicates from the outset that he would like to see one variable expressed in terms of the other and only then is he comfortable with "... for each value of x in the domain, you would get a value of y".

S: Solving in terms of either x or y would be kind of nasty.
I: To actually do it?
S: To actually do it. To find it. So I looked at it where you could pick out different values of x and different values of y that could make the equation to be true. Without having x to be a certain value and y always being a certain value. And that way, x and y could be manipulated as far as changing different values to make it, that expression, true.
I: I think I get the picture. You're saying that you've got these x and y values, right?
S: Correct.
I: And there may be lot's of them that you could change, but always so that they make the equation true. Okay, now how do you get a function from that?
S: Well for, so you could actually, probably, try to solve it as far as in terms of one of the variables and find that if you set a certain point, as far as domain, say if you solved for y, where the function would be in terms of x, y would equal something as far as x, various terms of x. And for each value of y which would be, for each value of x in the domain, you would get a value of y which would be the image of that.
I: How would you get that value of y? If you took a particular value of x, say 1 or $-.5$?
S: You would plug in the value into where x is and then solve the equation. And then you would come up with a value for y.
I: I see. And would you always be able to do that?
S: I don't think you are always going to be able to do it. I think there are restrictions as it is because you have, you always want it to equal 2. And you have $\sqrt{x}$ so you can't have a negative number. So there are certain restrictions you are going to have.
I: And in terms of the function terminology that we use, what would those restrictions be on?
S: Those would be restrictions on the range, I think. They'd have to be restrictions on the domain, as far as the numbers you could use, which would then, in turn, restrict your image.

It is responses such as this that suggest to us that an important ingredient of an action conception of function is the requirement of an

explicit formula for calculating a value of the dependent variable, given a value of the independent variable. Another indication of one of the aspects of our theory is this student's suggestion of restricting the domain which we take as possibly a reconstruction of knowledge (by redefining the situation) to fit his current mental scheme, in particular to ensure that he would have an expression to evaluate.

Indications of transition from action to process conception. In the following example, the student is struggling with her discomfort at the absence of a pattern (that could be used in an action), but may be resolving it by taking the more process-like view of a general input/output system.

S: ... you definitely have an x and a y value, x and an $f(x)$ value. I mean, I don't really know what ... There's really no pattern. I don't see any pattern but you definitely have an input and for that specific x value you have a y.

Sometimes, a student can learn from other students. The following example considers an expression which is a sequence of boolean values (Situation 10 in Appendix 1) so it may be interpreted as a boolean valued function of the positive integers. The student may be moving from an action to process conception or extending her process conception to a new situation.

I: You said "No, that's not a function, it's just an expression". What do you think about that now?
S: Some people in class brought up a very interesting point. A boolean expression could be a function, as far as the value going to be either true or false? And I never thought of a function in terms of being a boolean expression.
I: Okay, could you explain how, then, what's written there is a function? If you think now that it is a function.
S: Okay. Well, the domain of the function in this case would be just this is the number of integers from 1 to 100. And the image would be true and false. Whereas the process is from where an integer is converted to the corresponding value of true or false, depending on that boolean expression.
I: Do you want to read the boolean expression to me?
S: It's $2^n > (n^2 + 3n)$.
I: Okay, so what would be the value of this function at the point 2?
S: Well, it would be false because you have $4 > 10$.
I: And that's not true?
S: And that's not true.

Finally we see how a student revised his understanding of constant functions (Situation 6 in Appendix I) which can be particularly difficult. We can see here how the student grasps the idea of a constant function as a special kind of process.

S: I think I was thinking this is just like a graph of something. And that it was the equation $y = 5$. Or something like that.
I: Okay, and that's what you were thinking before, right?
S: Yes.
I: Okay. And that's why you wrote that. What about now? What do you think?
S: Well, let's say it is a function and it's just got the domain here be all positive integers, or all integers, or all numbers, really along the y-axis. And its range is just 5.
I: It's just 5. If that point happened to be at 5. Okay. And what would the process be?
S: The process would be just take a number and it would just be equal to 5 again.

Expressing a process conception. Here are some typical examples of students displaying process conceptions. In the first (Situation 10 in Appendix I) the student discusses the particular example in a manner which can indicate either an action or process conception, but in the end sums up the situations from a very clear process point of view.

S: Here you're, like in the other one you're taking values of n between 1 and 100. And you're going in and looking at, to see if this equation $2^n > n^2 + 3n$, to see if that by plugging in this n, if this equation is true or false.
I: Oh, okay. So what comes out of it then? So we've plugged it in . . .
S: This time you're not, this time you are going to get out like a true or false answer. You're not going to come up with a number. You're going to come up with a true or false. Whether that equation holds.
I: So again, you still have, from your definition then it fits because of what?
S: You have numbers being inputed. And a process being done. And you're going to come out with an answer. In this case being true or false.

Next, we have a student considering a situation which is a set of ordered pairs (Situation 1 in Appendix I). She shows how the computer has become a metaphor for her thinking about processes. What is more important is how she constructs the process which is definitely coming from her and not indicated in any way in the formal symbolism used to express the situation.

S: Yes, this is a function because it is a set of tuples. Your first, this is a set of tuples. Your first element of your tuple is going to be an integer, and if its the set of all integers, you're not going to have any repeats in there. Okay. And on your second element of the tuple it's just $2x + 1$.
I: Okay. So go back to your definition for me. So how does that satisfy your thought of what it needs to be?
S: Okay, when I look at these I think of putting them into the computer. Okay, I'm going to put this set into the computer. Then, like if I were going to call up like t(1), okay, I'm going to get out 3, is going to be my answer. I input the first element of the tuple, out comes the second, the second element of the tuple.
I: Okay, so the input you said was?
S: The first element of a tuple.
I: The first, what about that manipulating, that process you were talking about?
S: The process is you input the first element of a tuple. It looks at its tuples and finds one that had this first element in it, and outputs the second element.

It might be interesting to compare this last response with the results of Even (1988) who reports teachers' difficulty in seeing the process of a function when it is represented as a set of ordered pairs.

A tendency to respond. In Section 2 we referred to a person's mathematical knowledge as a *tendency* to respond. In the following excerpts we see a student who insists in one place that a process which gave the same result for two different inputs was not a function, but at another time in a mathematically similar situation, avoids this error completely. She begins with a good definition of a function.

S: A function is a system that when you put something in it gives a definite answer. And it would give you one answer as opposed to something that would, for the same system you put in, it would give you one answer as opposed to two or three answers.

But she denies that a boolean expression represents a function, not because of the lack of numbers in the output, but because of what amounts to a failure to be *1-1*.

S: No because this one, you're looking for true and falses.
I: Uh-huh.
S: It's a boolean expression here, so you'd, you'd either get true or false and so you could get true more than once or false more than once.

She repeats this error in several situations throughout a lengthy interview. At the end, however, she gives the following reaction while considering a table of dues owed by members of a club (Situation 20 in Appendix I). Notice that there is a key prompt by the interviewer who introduces the conflict between the definition and the result of applying it.

S: Well, this one I'm not sure because your function would be putting in the name and your output, this would be your domain and this would be your range but you've got two with 12's. So by my definition it wouldn't be!! I don't know if it is or not! I mean the actual . . .
I: It's all right.
S: I guess what I was thinking was that you would, the function, I don't know.
I: So you are seeing that there is a problem because of the names, they owe the same amount of dues.
S: Yea, so . . .
I: So what you are wanting to say. No this isn't a function because . . .
S: Yea.
I: . . . or are you thinking . . .
S: By the definition I've set up, it is not, because it has two numbers that are the same.
I: So do you feel confident in saying that or are you thinking more along the lines that there is a problem with your definition? I mean I don't want to put words in your mouth or anything, I just want to, I'm curious as to what you're thinking.
S: Um, well, just by looking at it, I would almost think that there would be a problem with my definition. Because it looks like it should be a function, where you put it in . . I don't know.

A little later after looking at a similar problem in which she says correctly that it is a function, the interviewer returns to the table and asks if it would be a function. The student says that it is a function, it is just not *1-1*. She then repeats her correct analysis for several examples. Although this might indeed be an illustration of a "tendency to respond", it is also possible that it represents an example of a student learning during the interview, which raises the question of the permanence of such knowledge.

Single-valued vs. 1-1. Confusion between the requirement for being a function and the definition of *1-1* is widespread (See, for example, Lovell (1971), Thomas (1969), Markovits et al. (1986), Leinhardt et al. (1990).) We have seen in the interviews exchanges which suggest that the student may be resolving it as a result of developing a process conception of function. Here is an example. First she discusses a situation (Situation 15 in Appendix I) that before instruction she had indicated was not a function, and is responding to a request for an explanation.

S: I'm not sure. I think I wasn't real sure when I did it because you can put in two different values for x, positive and negative, and still come up with the same value for y. And I didn't know if that was right or not.
I: Can you explain it by using some kind of number and explain it to me?
S: Well, if you put, if you say $x = 1$ then you're going to get $y = 1$ and when you say $x = -1$ you're going to get $y = 1$. But I think that can be right because if you have two points in the domain each one of those can have the same value.

And a little later with an example which is *not* single-valued, she deals with the issue of being undefined, but then the situation recurs. She shows some stability in not losing her grasp of the distinction between single-valued and *1-1* even though she has a general feeling of uncertainty.

S: Yes, but the same reason I said for the other one. I'm not quite sure, because if you put in a -1 you can't evaluate it. Because if $x = -1$, then $y^4 = -1$ and you can't take the 4th root of a -1. It will give you OM. So it's not a function then. I don't think. Well, unless you stipulate that the domain is positive. If the domain is positive then it is a function and you get positive integers for the range.
I: Okay, suppose $x = 1$. Okay?
S: Then it will return 1.
I: Would that be the only possible value for y?
S: Oh! It would be plus or minus 1, won't it? So it's not a function then because you have one domain going to two ranges.
I: Is that what you were thinking or what? Did you think about it?
S: I didn't really think in terms of domain and range when I did this. I was really kind of confused when I went through and looked at it.

FINAL EXAM

The responses on this instrument were also scored on the basis of 1 for an essentially correct answer and 0 otherwise. The average percentage score

for each question is given in Table VII. For Questions 2a, 2b, 3, 5a, and 5b the students maintained their high performance rates from the instrument Some Questions About Functions. There was some deterioration on Questions 2c, 4a, and 4b. Finally, on Question 5c the scores were very low. The overall average shows a small drop from the corresponding score on Table VI.

TABLE VII
Final Exam – percentage scores

Question 2a	83.9%
Question 2b	83.5
Question 2c	40
Question 3	87.2
Question 4a	54.5
Question 4b	47.5
Question 5a	75.8
Question 5b	73.7
Question 5c	30.9

We conclude with the comment that we tracked individual students through all of these observations and noted that of 59 students, 7 appeared to start the course with strong process conceptions, 24 showed clear progress throughout the semester that seemed to be more than one might expect from ordinary instruction and the other students made only a small amount of progress or their performances appeared to oscillate. None of the students got noticeably worse.

6. CONCLUSIONS

6.1. *Interpretations*

Comparing Tables I and III suggests that, after some time in a general computer environment, students tended to move from Prefunction through Action to Process in their conception of function. As we have indicated, their success with Some Questions on Functions is consistent with the suggestion that, as a result of the instructional treatment on functions, they tended to develop and strengthen their process conception of function. In the interviews about Functions in Situations, many students described how they constructed mental processes, not explicit in the given Situation, in order to use a function to model the situation. We consider that these results indicate that the instructional treatment had a certain amount of success in helping students construct a process conception of function. In the Final Exam, there is no mention, on any of the questions, of anything connected with ISETL, programming, or computers. It is entirely a mathematics examination and one on which many instructors would be, as we were, proud to have their students perform well. The students' success on this exam supports our theoretical contention that the ability to construct a process conception of function can lead to significant improvement in mathematical performance. In this sense, the results of this paper lend credence to our theoretical perspective. We can offer now some interpretations of specific results.

Tables 3 and 4b. We noted that the students showed a little more in the way of a process conception by their explanations than by their examples. This seems a little unusual, and was the reverse of what happened relative to Tables I and IIb. People in general tend to do better in producing something specific than in understanding what they are doing (Piaget, 1978). We can offer some conjectures as to why. There are other, equally plausible possiblities. One possible explanation is that success with examples is higher than understanding when one is learning procedurally or instrumentally (in the sense of Skemp, 1976) and solely "from experience". It could be, however, that when a person is actively constructing concepts at a higher level of thinking, the relationship between success and understanding is quite different. A suggestion that is consistent with this is that in Category I which makes the major contribution to the first column in Table IVb, most of the "non-function" responses were unsuccessful attempts to use ISETL to describe a process. Perhaps these students are in the midst of trying to construct a process conception of function in order to make sense of ISETL experiences. One might tentatively conclude from this observation that a process conception of function was in the students' *proximal zone of knowledge* in the sense of Vygotsky (1978).

Tables 2b and 4b. There is an anomaly here in that the number of responses in the first column of Table IVb is higher than in the first column of Table IIb. Otherwise, the data is roughly consistent with the decrease of pre-function responses in favor of action responses that we saw in the responses to "What Is A Funtion" (comparison of Tables I and III). The anomaly could be explained by the fact that, unlike the other three observations, in this case the students were asked for three examples and so an omission could indicate that the student could only think of one kind of example.

Final Exam, Question 5. The success with the first two parts of this question suggests that many students were making progress in developing an object conception of function at least to the extent of understanding actions and processes that acted on functions and produced functions as answers. The result on the last part of this question shows that almost a third of the class was able to reverse such a process.

6.2. *Improvements in the Process Conception*

The students made significant progress in the development of their process conception, and they showed this in several different ways in different observations. In the beginning, at least half of the students did not indicate by their response to What Is A Function (Table I) that a process was an essential part of a function. Their examples were, for the most part, restricted to algebraic or trigonometric expressions (Table IIb). Even after some time in a general computer environment, although the examples they offered became a little richer, they still did not involve processes very much (Table IVb). The tabulation in Table IVa continues to suggest that they either did not know what a function was or that it was mainly an action for them. This impression that the students were working with functions as actions that are not yet interiorized into processes is strengthened by their analyses of the Functions in Situations. Looking at Table V, we see that only in the case of a func where the action of the function is explicit, did a high percentage of students see a function. Not nearly as many saw functions in tables, tuples and smaps. Here the action is not so explicit and must be constructed by the student (which means that it will have been interiorized, since it is constructed in the student's mind from the beginning). Finally, in the case of strings, equations, graphs and physical situations, where even more is required of the mental constructions, less than one third saw functions. An interesting side observation to make here is that the results of Table V suggest that students are more likely to see a function when they can think of it as being on a computer than when it is only written. They seem to be particularly unlikely to think of a function when presented with a graph, especially if there is anything unusual about it.

By the time our instructional treatment was completed, the students were able to solve quite difficult problems that seem to require them to construct function processes in their minds and work with these processes in fairly complicated ways. The scores in Table VI are uniformly high, and Table VII shows that they performed reasonably well on some very difficult questions. Because of the fact that the problems on the two instruments, Some Questions About Functions and Final Exam were not only difficult but were, in many cases, new and unfamiliar for the students, it seems that their understanding may have some robustness and is not immediately destroyed by simply being in a difficult situation. The interviews suggest that these scores represent more than just success with specific problems. One can see explicitly how a student uses a process to describe the function in the case of a set of ordered pairs. Many of them move from thinking that functions can only work with numbers to considering more general objects (e.g., boolean values) in the domains and ranges of their functions. The construction of a boolean valued function of two variables in the case of an equation was a striking spontaneous invention of several students during classroom discussions. Another interesting thing that can be seen in the interviews is the struggle that students go through in reconstructing previous knowledge to deal with new situations. In some cases, this seems to take place right in the interview. It suggests that the knowledge of these students really is coming from construction and growth, rather than merely regurgitating something they have heard in class or repeating actions on which they have been drilled.

6.3. *What Caused the Improvement*

There appears to be little question that significant improvement in understanding functions occurred during the course for a large number of the students. Much less clear is the reason why this improvement occurred. We feel that there are some indications that progress occurred because of the particular instructional treatment that we used. The students showed, in the examples they offered, that their experience with computers was having an effect on their thinking. In the interviews we can point to several instances in which one could strongly infer that the kind of constructions our theoretical analysis describes were indeed taking place in connection with computer metaphors that students were offering spontaneously. The apparent growth in conception that we saw in some interviews is in accord with our model of mathematical knowledge, as described in Section 2. We could often see what can be taken to be aspects of this model in action. The struggle in which students engaged during the interviews can be interpreted as strain in their efforts to construct or reconstruct their conception of function in order to view these situations as functions. They had a high degree of success (after instruction) in interpreting situations as functions and this, in comparison with their early responses, indicates that at some time (after the beginning of the course) they developed some of the building blocks of the function concept. We suggest that this happened as

a result of the instructional treatment. This is certainly borne out by our impressions during classroom discussions. Often, in the interviews, we saw students echoing the very ideas that were discussed in class. Although this might be taken as simply parroting back what they had heard, we don't think so because of the ability they demonstrated, in all of the post-treatment observations, to apply those ideas to situations that were very different from the context in which the ideas originally arose.

6.4. *Revisions of the Epistemology*

According to our research paradigm, each study that attempts to explain results in terms of our epistemological analysis of concepts leads to a combination of assimilation of the results to the epistemology and changes in the epistemology in order to accommodate the results. Up to this point in the paper we have tried to show how our epistemological analysis can be used to explain the results. Now we consider what changes we might make in our epistemology to increase our ability to explain, perhaps predict, but most important to design effective instructional treatment. The main area in which we can improve our analysis is in the meaning of the reflective abstraction which consists of constructing a process. We did not find that our language in discussing processes and actions was completely adequate once we went beyond simple issues like What Is A Function and Examples Of Functions. Let us try to give now, as a result of the insights gained from the present work, a fuller statement of what we mean by action and process. We refer the reader to our initial descriptions of action and process at the end of Section 2. We would now add the following:

Both actions and processes transform objects. The main difference (which we saw quite explicitly in the interviews) between an action and a process is the need in the former for an explicit recipe or formula that describes the transformation. Moreover, in an action one tends to think about the transformation in a step-by-step manner with the steps related only by the recipe, and not by any relationships that exist in the mind of the subject. A process, on the other hand, represents a transformation that does not have to be very explicit, nor must a subject be absolutely certain that it exists. It is only necessary that the transformation be imagined, in the mind of the subject, as a more or less certain possibility. Thus we might summarize by saying that in the case of an action the transformation is relatively external to the thinking of the subject, whereas in a process it is more internal. (We say "relatively" because in this constructivist epistemology, everything is more or less internal to the subject.)

There are certain relatively indirect indicators that an individual has constructed a process in a certain situation relative to functions. One is that the subject is not restricted to numbers for the domains and ranges of the functions. Another is that he or she is able to imagine certain operations with functions (even if there are no explicit formulas) such as adding or composing two functions, or reversing a function. Finally, an important way in which one can infer something about the nature of a person's process conception of functions is to see what happens when, in dealing with functions as objects (e.g., in thinking about the composition of two functions or in a problem such as Question 5 on the Final Exam), it is necessary to de-encapsulate the objects and re-present their processes. If this comes out as a not necessarily specific transformation of objects, then the subject's conception might have been a reasonable process. If it is not possible, or requires explicit formulas, then the subject may not have gone beyond actions.

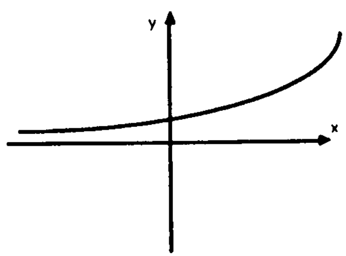
The distinction between actions and processes and the corresponding external and internal nature of understanding still requires more research and is the subject of ongoing studies which will be described in future reports.

APPENDICES

I. *Functions in Situations*

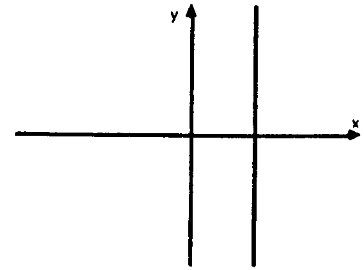
Directions. Look at each of the following situations and decide if it is possible to use one or more functions to describe it. If yes, then describe your function(s) briefly. If no, then explain why not.

1. $\{[x, 2x+1]$: x in the set of all integers$\}$
2. $[2^n + n^3$: n in $[1\,..\,100]]$
3.



4. $y^4 = x^2$
5. "Purdue Women's Basketball Team Wins NCAA"

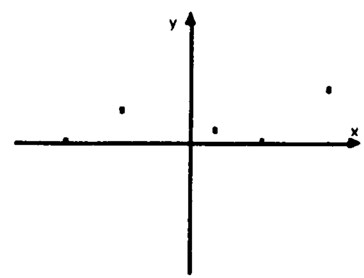
6.



7.

```
R:=func(x);
  if is_number(x) then
    if x > 0 then return x**2+1;
    elseif x=0 then return 1;
    else return x+1;
    end;
  end;
end;
```

8. $\{[1, 2x]$: x in $\{1\,..\,100\}\}$
9. $x^2 + 3x + 2 = 0$
10. $[2^n > n^2 + 3n$: n in $[1\,..\,100]]$
11. A square in the plane centered at the origin is rotated clockwise by 90°.
12. A record of all NCAA Division I men's basketball teams giving, for the 1987–88 season, each team's field goal shooting percentage at home and its field goal shooting percentage away.
13. $2x^3y - \sqrt{x}\log y = 2$
14. $\{[x^2, x^2]$: x in $\{1\,..\,100\}\}$
15. $y^4 = x^3$
16.



17. $\{[x, y]$: x, y in the set of all rational numbers$\}$

18.
$$\left.\begin{aligned} x &= t^3 + t \\ y &= 1 - 3t + 2t^4 \end{aligned}\right\} t \text{ a real number.}$$

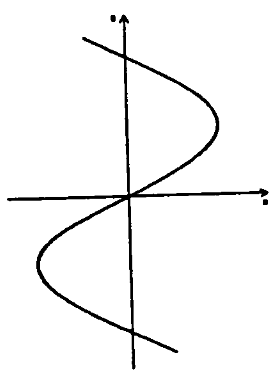
19. A swimmer starts from shore and swims to the other side of the lake.
20. The club members' dues status.

Name	Owed
Sue	$17
John	6
Sam	27
Bill	0
Iris	6
Eve	12
Henry	14
Louis	6
Jane	12

21. "ELPRHZAUPQDRMW"
22.

```
Q:=func(x);
  if is_integer(x)
    then return
    23+random(abs(x));
  end;
end;
```

23. $\{2n + \text{random}(9)\colon n \text{ in } [1\,..\,50]\}$
24.



II. *Some Questions about Functions*

1. One can sometimes tell a great deal about a text (its author, when it was written, etc.) by various characteristics (e.g. word length, frequency of occurrence of certain words, etc.).

For example, one might use the number of occurrences of each vowel a, e, i, o, u in a text. Suppose that you were trying to analyze the following text (of course, in actual practice, one uses a much larger amount of material) by studying the number of occurrences of each vowel.

The quick fox jumped over Farmer Brown's lazy dog.

Explain how you might use a function to organize this information. Write whatever you think is necessary about your function to specify it. Evaluate your function once.

2. Suppose that int_str is an ISETL representation of a function that takes a positive integer and converts it to a string which is its name. Thus, one could have the following terminal session.

```
int_str(237);
"two hundred and thirty seven";
```

Now consider the following ISETL func which represents a function F.

```
F:=func(n);
     if is_integer(n) and n>0
       then return
       {[x, int_str(x**n)] : x in [1..10]};
     end;
   end;
```

(a) What is the value of $F(3)(2)(4)$? Explain how you got it.
(b) Find a such that $F(4)(3)(a) =$ "t".
(c) Find b such that $F(2)(b)(1) =$ "o".
(d) Find c such that $F(c)(1)(2) =$ "n".

III. *Final Exam Questions*

2. In each of the following three questions, f, g, h are functions whose domains and ranges are the set of all real numbers, and such that $h = f \circ g$.
   (a) If only the information in the following table were known, would it be possible to find $h(0)$? If so, find it and if not explain why not.

x	$f(x)$	$g(x)$
-1	2	-3
0	-3	-1
4	1	2

   (b) If only the information in the following table were known, would it be possible to find $f(2)$? If so, find it and if not explain why not.

x	$h(x)$	$g(x)$
-1	1	-3
4	π	1
π	0	2

(c) If only the information in the following table were known, would it be possible to find $g(4)$? If so, find it and if not explain why not.

x	$h(x)$	$f(x)$
-1	1	-2
2	3	1
4	-2	π

3. Let the two functions F and G be defined as follows.

$$F(x) = \begin{cases} \dfrac{x}{x^2+1} & \text{if } x < -2 \\ 2x+3 & \text{if } -2 \leqslant x < \frac{1}{2} \\ \sin x & \text{if } \frac{1}{2} \leqslant x < 1 \\ \sqrt{x} & \text{if } 1 \leqslant x \end{cases}$$

$$G(x) = \begin{cases} -4 & \text{if } x \leqslant 0 \\ 2x & \text{if } 0 < x \leqslant 1 \\ \sqrt{x} & \text{if } 1 < x \end{cases}$$

Find the product $F \cdot G$ of the two functions.

4. Let f, g be two functions whose domains and ranges are subsets of the set of real numbers. Prove or find a counter-example to the following two statements.
   (a) If f, g are both *1-1* then it follows that $f+g$ is *1-1*.
   (b) If f, g are both onto then it follows that $f+g$ is onto.
5. Let $\mathscr{F}$ be the set of all functions whose domain and range are the set of all real numbers. Let **D** be the operation that acts on a function, say f, in $\mathscr{F}$ and transforms it to the function f' (the derivative). Let **K** be the operation that acts on a function, say f, in $\mathscr{F}$ and transforms it to the function h where $h(x) = f(-2x)$.
   (a) Are **D** and **K** functions? Explain. If not, can you change them a little so that they are?
   (b) If f is the function in $\mathscr{F}$ defined by $f(x) = x^3$, what is $\mathbf{K}(f)(3)$?
   (c) Describe the inverse of **K**.

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