Chapter 6. Reflective Abstraction in Advanced Mathematical Thinking

Ed Dubinsky

Our purpose in this chapter is to propose that the concept of reflective abstraction can be a powerful tool in the study of advanced mathematical thinking, that it can provide a theoretical basis that supports and contributes to our understanding of what this thinking is and how we can help students develop the ability to engage in it. To make such a case completely, it would be necessary to do at least four things: explain exactly what we mean by reflective abstraction; show how it can be used to describe the epistemology of various mathematics concepts; indicate how it can suggest explanations of some of the difficulties that students have with many of these concepts; and establish that it can influence the design of instruction in ways that result in a significant improvement in the extent to which students appear to acquire these concepts.

We are certainly not ready to do an exhaustive job on all four of these tasks. Indeed, our main concern here is to make some progress with the first two. There will be a few examples of the third, and we will make reference to other papers in which we have made a start on the fourth especially involving the use of computer activities to help students make mental constructions, with results that are encouraging.

Reflective abstraction is a concept introduced by Piaget to describe the construction of logico-mathematical structures by an individual during the course of cognitive development. Two important observations that Piaget made are first that reflective abstraction has no absolute beginning but is present at the very earliest ages in the coordination of sensori-motor structures [2, pp. 203-208][1], and second, that it continues on up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered as an example of the process of reflective abstraction [26, pp. 149-150].

In the majority of his own work, however, Piaget concentrated on the development of mathematical knowledge at the early ages, rarely going beyond adolescence. What we feel is exciting is that, as he suggested, this same approach can be extended to more advanced topics going into undergraduate mathematics and beyond. It seems that it is possible not only to discuss and conjecture but to provide evidence suggesting that concepts such as mathematical induction, propositional and predicate calculus, functions as processes and objects, linear independence, topological spaces, duality of vector spaces, duality of topological vector spaces, and even category theory can be analyzed in terms of extensions of the same notions that Piaget used to describe childrens' construction of concepts such as arithmetic, proportion, and simple measurement.

This is a strong claim embodied in the phrase "can be analyzed" and, before going further, it is necessary to explain exactly what we mean by it. The result of our discussion of reflective abstraction in this paper will be a general framework which can be used, in principle, to describe any mathematical concept together with its acquisition. We refer to this as the *general theory* but it does not, by itself, lead to any particular description. The investigator needs to make use of her or his *understanding of the mathematics*. Together these two are enough to obtain a description of any concept but the result would be far too *ex post facto* to expect it to have any relation to how students actually

[1] Piaget repeated many of his comments regarding reflective abstraction in several places, but was quite consistent on this topic. Hence, the references we give should be taken as representative.

might go about constructing the concept. It can serve as a guide to initial interaction with students, (which becomes more necessary as the concepts become more advanced), but we believe that an ess in the study of any concept is a long drawn-out, time consuming effort of *observation of student* construct mathematical concepts in order to make sense out of situations in which they find themsel but not necessarily, as the result of activites of a teacher). In summary we believe that although t particular concept can begin with a combination of the general theory and our understanding of th before very long the investigation should include observations of students and the analysis should from a synthesis of all three sources.

Our discussion of the specific concepts that appear in this paper reflects the "in-progress" nat Our study of mathematical induction and quantification has progressed quite far and what we say ab is really based on the kind of synthesis we suggest. The details of how we used these three sources conclusions mentioned here is fully laid out in the papers we have written on these topics [9,10,1 study of the function concept, we have made some observations of students so this has some effect o but the work is only beginning and the stronger influence comes from the general theory and ou of the mathematics involved. Because none of our material on functions has appeared elsewhere, the details here in Section 3.3 In all other topics mentioned in the paper, we have not yet begun observations so the discussion is based almost entirely on the other two sources and must be consid speculative. One of the purposes for including it is to suggest that the general theory can be used a start on studying just about any concept in mathematics at whatever level.

The starting point of our general theory is Piaget's notion of reflective abstraction. Unfortunat simple idea clearly explained in one place, but rather something that Piaget appeared to work with ov of time after he completed his empirical studies of children in development. It is important, howeve with a solid understanding of what he meant by it before trying to extend it to a wider class of math Therefore we begin this chapter with a section that gives a brief summary of this concept as Piaget a number of books and papers, mostly written in the last 15 years of his life. We will emphasize t aspects of reflective abstraction because these are the most important for the development of mathe during adolescence and beyond.

In the second section we will show how Piaget's ideas can be extended and reorganized to form a g mathematical knowledge and its acquisition which is applicable to those mathematical ideas that be the post-secondary level and continue to be constructed in the course of mathematical and other sc It is here, in Section 2 that we relate various aspects of the theory to specific topics in advance thinking and give several examples of how the general theory of reflective abstraction can suggest student difficulties.

Our analysis of a particular mathematical concept leads to what we call a *genetic decompositio* which is a description, in terms of our theory, and based on empirical data, of the mathematics inv subject might make the constructions that would lead to an understanding of it (which, in our thec different). It is important to note that we do not suggest that a concept has a unique genetic decomposition or that this is the way every subject will learn it. We only claim that observations of learning in progress form an important source for our genetic decompositions and we offer them as a guide for one possible way of designing instruction. In Section 3 we present genetic decompositions for three topics: mathematical induction, predicate calculus, and function, insofar as we have constructed them. The references given in Section 3 contain more information about examples of instructional treatments based on these genetic decompositions, using computer experiences, and about the generally encouraging results of implementing these treatments.

Finally, in Section 4 we discuss some of the educational implications of our theory of knowledge and learning and give an overview of how we go about designing an instructional treatment based on it. We feel that the material in this section is very much akin to the ideas in [36].

In writing this chapter we are trying to speak to two audiences: research mathematicians who also teach mathematics at the undergraduate level, and researchers in mathematics education who are interested in post-secondary topics. To the mathematicians we are trying to say that here is an example of an investigation in mathematics education research that is far from trivial, is taking place over a long period of time, is "solving" some problems, and is raising a lot of interesting questions. It will not be so easy to understand the ideas of a thinker like Piaget, but the effort is worth it and can make a significant contribution to your understanding of how and why your students do or do not learn, and the kinds of things you might be able to do about it.

To the mathematics educator we are trying to say that, although advanced mathematics topics need to be expressed in a formal language, the content of the ideas is far from formal. When the material is really understood, then behind every static set of symbols that is written down, there is, for the subject, a lively dynamic motion of processes, formation of objects, unpacking processes and hooking them together, transforming objects, going back and forth over processes, and so on through a rich world that is really there in the mind of the mathematician and will turn our students on if we can ever get them to touch it.

To both audiences we are trying to say that those other folks have something to offer and you should try to understand what is going on their heads. Indeed, we feel that if these two groups could be brought together with a higher degree of interaction than is found at present, then we could make a lot of progress together on some of the very serious problems of learning mathematics that are of major concern in the world today. We hope that this chapter makes a contribution to fostering such a synthesis.

1 Piaget's notion of reflective abstraction

The importance of reflective abstraction

Piaget distinguished three major kinds of abstraction. *Empirical abstraction* derives knowledge from the properties of objects [2, pp. 188-189]. We interpret this to mean that it has to do with experiences that appear to the subject to be external. The knowledge of these properties is, however, internal and is the result of constructions made internally by the subject. According to Piaget, this kind of abstraction leads to the extraction of common properties of objects and extensional generalizations, that is, the passage from "some" to "all", from the specific to the general

[14, p. 299]). We might think, for example of the color of an object, or its weight. It might be considered that these properties reside entirely in the object, but in fact, one can only have knowledge of these properties by doing something (looking at the object in a certain light, hefting it) and different individuals under different conditions might come to different conclusions about these properties.

Pseudo-empirical abstraction is intermediate betweeen empirical and reflective abstraction and teases out properties that the actions of the subject have introduced into objects [26, pp. 18-19]. Consider, for example the observation of a 1-1 correspondence between two sets of objects which the subject has placed in alignment [26, p. 39]. Knowledge of this situation may be considered empirical because it has to do with the objects, but it is their configuration in space and relationships to which this leads that are of concern and these are due to the actions of the subject. Again, of course, understanding that there is a 1-1 relation between these two sets is the result of internal constructions made by the subject.

Finally, *reflective abstraction* is drawn from what Piaget called the *general coordinations* of actions and, as such, its source is the subject and it is completely endogenous [23, pp. 89-97]. We will see many instances of reflective abstraction. but a very early example we can mention now is seriation, in which the child performs several individual actions of forming pairs, triples, etc., and then interiorizes and coordinates the actions to form a total ordering [20, pp. 37-38]. This kind of abstraction leads to a very different sort of generalization which is constructive and results in "new syntheses in midst of which particular laws acquire new meaning" [14, p. 299]. An example of this would be the concept of Euclidean ring which is certainly an abstraction and generalization. It might be considered, however, to derive from the properties of a *single* example — the integers.

We can see, therefore, that these different kinds of abstraction are not completely independent. The actions that lead to pseudo-empirical and reflective abstraction are performed on objects whose properties the subject only comes to know through empirical abstraction. On the other hand, empirical abstraction is only made possible through assimilation schemas which were constructed by reflective abstraction [26, pp. 18-19]. Consider, for example a physics experiment which may have the purpose of making an empirical abstraction to obtain factual data about a certain object. The experiment presupposes, however, an enormous range of logico-mathematical preliminaries — in deciding how to pose the question, in the construction of apparatus for "indirect observations" (e.g., triangulation to obtain distances between stars), in the use of particular forms of measurement, and finally, in setting out the results in logico-mathematical language. All of these are concepts that must have been constructed using reflective abstraction. [23,p. 91]. This mutual interdependence can be roughly summarized as follows. Empirical and pseudo-empirical abstraction draws knowledge from objects by performing (or imagining) actions on them. Reflective abstraction interiorizes and coordinates these actions to form new actions and, ultimately new objects (which may no longer be physical but rather mathematical such as a function or a group). Empirical abstraction then extracts data from these new objects through mental actions on them, and so on. This feedback system will be reflected in our extension of these ideas in the next section.

In empirical abstraction the subject observes a number of objects and abstracts a common property. Pseudo-empirical abstraction proceeds in the same way, after actions have been performed on the object. Reflective abstraction, however, is much more complicated. This is not surprising since, according to Piaget, "The development of cognitive structures is due to reflective abstraction..." [26, p. 143]. Before going into the nature of this fundamental process, therefore, we should say a few things about its importance, in Piaget's view, to cognitive thought in general and mathematics in particular.

Piaget considered that describing reflective abstraction was one of the two central problems of genetic epistemology (self-regulation or equilibration being the other) [21, p.78]. It is, for him, the motive force of the the reconstructions involved in the passage through the stages of sensori-motor actions, semiotic representations, concrete operations and formal operations [2, p.245] and furnishes the increasingly complex materials for construction of structures [18, p.62]. He felt that a subject's consciousness of physical or mental actions is one of the results of applying reflective abstraction to them [2, p. 282] and that this is how one comes to understand these actions [24, p. 300].

In two books [24,25] Piaget interpreted the results of many experiments with children in terms of reflective abstraction. But its role is not restricted to the intellectual development of children. From Piaget's psychological viewpoint, new mathematical constructions proceed by reflective abstraction [2, p.205]. Indeed, he considered it to be the method by which all logico-mathematical structures are derived [17, p. 342]; and that "it alone supports and animates the immense edifice of logico-mathematical construction" [23, p.92].

In support of his position on the role of reflective abstraction in advanced mathematical thinking Piaget tried to explain a number of major mathematical concepts in terms of the constructions that result from this psychological process. These included the idea of Gödel's incompleteness theorem [2, p.275], the abstract concept of groups [18, p. 19], Bourbaki's attempts to encompass all of mathematics within three "mother structures" [18, p.24], the general theory of categories [18, p.28], the impossibility of constructing the set of all sets [18, pp.70-71], and the mathematical concept of function [19, p.168]. More generally, Piaget considered that it is reflective abstraction in its most advanced form that leads to the kind of mathematical thinking by which form or process is separated from content and that processess themselves are converted, in the mind of the mathematician, to objects of content [20, pp.63-64 and pp. 70-71].

An important sidelight to this line of thinking is Piaget's suggestion that the move to abstraction, the axiomatization of modern mathematics is, in fact, a continuation of many other kinds of transcendence in human thought. He was of the opinion that axiomatic reorganization and formalization are "not opposed to nature but on the contrary appear as 'natural' as the pre-axiomatic constructions" [2, p.134]. If this were so, it would have important implications for mathematics education, at least at the post-secondary level. It suggests that not all pedagogical wisdom is contained in the admonition to "relate abstract ideas to their concrete manifestations in everyday life". Aside from the fact that this is not always possible (although computers have widened its scope), one might try to argue from Piaget's observations that since abstract thinking is a natural development of the human mind, a continuation of a path on which everyone begins, it may be possible to excite and motivate students, not only with "real-world" applications but with the intellectual ideas and constructions of abstract mathematics. Of course, any such attempt must deal with the pervasive lack of success that mathematics instruction has had in this direction. One of the purposes of our research is to change that through a greater understanding of the nature of abstract mathematical thought and how it develops.

Returning to the ideas of Piaget, it is important to emphasize that there is no suggestion here that all (or any) of the advanced mathematics described above is actually done by any kind of direct application (conscious or otherwise) of reflective abstraction. This was not Piaget's purpose in trying to analyze that aspect of thinking. The point, rather, is that when properly understood, reflective abstraction appears as a description of the mechanism of the development of intellectual thought. It is important for Piaget's theory that this same process that describes advanced mathematical thinking appears in cognitive development throughout life from the child's very first coordinations that lead to concepts such as number, measurement, multiplication, and proportion [20, pp. 70-71]. An important ingredient of Piaget's general theory (on which he worked for 60 years) that relates biological evolution to the development of intelligence is the idea that reflective abstraction is one isolated case of certain very general processes that are found throughout living creation [17, p. 331]. He tried to establish an analogy between the development of intelligence on the one hand where phenomena are internalized to a lesser or greater degree, the former leading to assimilation of these structures by means of existing cognitive schema and the latter resulting in accommodation and the reconstruction of mental structures; and evolution on the other hand where the lesser degree of internalization leads to the phenotype and the greater results in the genotype and evolutionary advances.

The nature of reflective abstraction

As we have seen, reflective abstraction differs from empirical abstraction in that it deals with action as opposed to objects and it differs from pseudo-empirical abstraction in that it is concerned, not so much with the actions themselves, but with the interrelationships among actions, which Piaget called "general coordinations" [24, p. 300].

According to Piaget, the first part of reflective abstraction consists of drawing properties from mental or physical actions at a particular level of thought [2, pp. 188-189]. This involves, amongst other things, a measure of cognizance or consciousness of the actions [17, p.320]. It can also include the act of separating a form from its content [20, pp. 63-64]. Whatever is thus "abstracted" is projected onto a higher plane of thought [26, pp. 29-31] where other actions are present as well as more powerful modes of thought.

It is at this point that the real power of reflective abstraction comes in for, as Piaget observes, one must do more than dissociate properties from those which will be ignored or separate a form from its content [27, p. 206]. There is "a process which will become increasingly evident over time: the construction of new combinations by a conjunction of abstractions" [20, p. 23].

Piaget seemed to feel that this construction aspect of reflective abstraction is more important than the abstraction (or extraction) aspect [20, p. 64]. Not only did he assert, as we observed earlier, that construction of this kind is the essence of mathematical development, and that combining formal structures is a natural extension of the development of thought [20, p. 64], but he also used his analysis of this process to deal with the philosophical question of the nature of mathematical thought ([2]).

Certainly for our purposes, the construction aspect of reflective abstraction will play the major role.

Examples of reflective abstraction in children's thinking

We begin with some of Piaget's examples of reflective abstraction in logico-mathematical thinking at the earlier ages. This is important because of his insistence on the continuity of development as part of his search for a single process or set of processes that related to biological development as well as intellectual development [17, p. 331]. Our suggestion in this chapter is that the specific construction processes that can be used to build sophisticated mathematical structures can be found, already, in the thinking of young children.

commutativity of addition. The discovery that the number of objects in a collection is independent of the order in which the objects are placed requires first that the child count the objects, reorder them, count them again, reorder and count, etc. Each of these actions are interiorized and represented internally in some manner so that the child can reflect on them, compare them, and realize that they all give the same result ([21, pp. 16-17]).

number. According to Piaget [16], the concept of number is constructed by coordinating the two schemas of classification (construction of a set in which the elements are units, indistinguishable from each other) and seriation (which, as we observed earlier, is itself a coordination of the various actions of pairing, tripling, etc.).

measurement. This is obtained as a coordination of subdivision (into units) and displacement of the unit as many times as necessary to match the object being measured [28].

trajectory. The traversal of a path is understood as a coordination of successive displacements to form a continuous whole [23, p. 90].

see-saw. The balancing of objects on two sides of a see-saw by a combination of actions on *both* sides involves more than just keeping two things in mind at the same time. Because he observed a considerable delay between the time that a child could create the balance and the time that the child appeared to understand how he or she had done it, Piaget saw this as a coordination of two actions into a single system [25, p. 96].

multiplication. Both psychologically and mathematically, multiplication is the addition of additions. It is, however, objects that are added in the sense that addition is an operation applied to something. In order, therefore, to multiply, it is necessary first to encapsulate the (mental) action of addition into an object (or set of objects) to which addition can be applied ([26, p. 31]).

proportions. Again, proportion is obtained as a relation between relations, which latter must first be encapsulated into objects to be compared [20, p. 71].

fluid levels. In an experiment asking children to predict the level to which a known amount of fluid would rise in a vessel with sloping sides and markings at equal height divisions [19, Ch. 7], Piaget pointed out that this situation is a case of "variation of variations". That is, the differential in two vertical markings is a variation, but the amount of change also varies because of the sloping sides. Hence, the first variation must become an object to which an action is applied (sloping sides) resulting in a higher order variation.

In each of these examples we have emphasized the construction aspect of reflective abstraction. Piaget was somewhat more vague and less consistent on the aspect of projection onto a higher level of thought. Sometimes it was just a question of a more powerful schema such as full seriation coming from pairs [20, p. 37], or, in general, the fact that a reconstructed schema was sure to be richer because it coordinated several schemas [27, p. 206]. If a physical action was interiorized it was possible to reason about it, which Piaget took to be a higher level than just acting [23, p. 89]. Being conscious of a thought was considered to be on a higher plane [2, p. 189], as was treating processes on lower levels as objects and acting on them [20, p. 70]. Two quite disparate examples which Piaget gave was that of G. Cantor who elevated the primitive notion of 1-1 correspondence to a tool for studying infinity [2. p. 242] and the hierarchy of levels in his own notion of stages of development.

As indicated earlier, we will not pursue this question of projection in this chapter but concentrate on construction.

Various kinds of constructions in reflective abstraction

In considering the above examples of reflective abstraction as methods of construction, we can isolate four different kinds which will be important for advanced mathematical thinking. We add a fifth which Piaget considers at length, but was not, for him, part of reflective abstraction.

- With the appearance of the the semiotic function, that is, the ability to use symbols, language, pictures, and mental images, the child performs reflective abstractions to represent [18, p. 64], that is, to construct internal processes as a way of making sense out of perceived phenomena. Piaget called this *interiorization* [23, p. 90] and referred to it as "translating a succession of material actions into a system of interiorized operations" [2, p. 206]. The commutativity of addition described above is one example of this. (See also [36, p. 197].)

- Several of our examples such as measurement, trajectory, and see-saw involve the composition or *coordination* of two or more processes to construct a new one. This is to be distinguished from Piaget's phrase, "general coordinations of actions" which refers to all ways of using one or more actions to construct new actions or objects.

- Multiplication, proportion and variation of variation exemplify the construction which is perhaps the most important (for mathematics) and most difficult (for students). This is *encapsulation* or conversion of a (dynamic) process into a (static) object. As Piaget put it, "...actions or operations become thematized objects of thought or assimilation" [26, p. 149]. He considered that "The whole of mathematics may therefore be thought of in terms of the construction of structures,...mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures" [20, p. 70]. From a philosophical point of view, Piaget was applying the idea of encapsulation to the relativity between form and content when he referred to "...building new forms that bear on previous forms and include them as contents" and "reflective abstractions that draw from more elementary forms the elements used to construct new forms" [26, p. 140].

- When a subject learns to apply an existing schema to a wider collection of phenomena, then we say that the schema has been *generalized*. This can occur because the subject becomes aware of the wider applicability or when a process becomes an object as, for example, the ratio of two quantities, or addition, so that a schema such as equality or addition again can be applied to it to obtain, respectively, proportion or multiplication. The schema remains the same, the only change is in the object that can now be assimilated by the extended schema. Piaget referred to this as a reproductive or generalizing assimilation [20, p. 23], and he called the generalization *extensional* [14, p. 299].

- Once a process exists internally, it is possible for the subject to think of it in reverse, not necessarily in the sense of undoing it, but as a means of constructing a new process which consists of *reversing* the original process. Piaget did not discuss this in the context of reflective abstraction, but rather in terms of the INRC group [27, p.205]. We include it as an additional form of construction.

2 A Theory of the Development of Concepts in Advanced Mathematical Thinking

Objects, Processes, and Schemas

Although, as we have pointed out, Piaget believed that reflective abstraction was as important for higher mathematics as it was for children's logical thinking, his research was mainly concerned with the latter. In order to try to develop the notion of reflective abstraction for advanced mathematical thinking, we will isolate what seem to be the essential features of reflective abstraction, reflect on their role in higher mathematics, and reorganize or reconstruct them to form a coherent theory of mathematical knowledge and its construction. It will not escape the reader's notice that we are attempting to perform a reflective abstraction on reflective abstraction itself.

For us, reflective abstraction will be the construction of mental objects and of mental actions on these objects. In order to elaborate our theory and relate it to specific concepts in mathematics, we will use the notion of *schema*. A schema is a more or less coherent collection of objects and processes. (This is slightly different from Piaget's term *schème*) A subject's tendency to invoke a schema in order to understand, deal with, organize, or make sense out of a perceived situation is her or his knowledge of an individual concept in mathematics. Thus an individual will have a vast array of schemas. There will be schemas for situations involving number, arithmetic, set formation, function, proposition, quantification, proof by induction, and so on throughout all of the subject's mathematical knowledge. Obviously, these schemas must be interrelated in a large, complex organization. For example, there will be a proof schema, which can include a schema for proof by induction. This latter in turn could include a schema for proposition valued functions of the positive integers (see p. 19). Hence there would be a relation with the schemas for number, for function, and for proposition. On the other hand, there is a sense in which a proof is an action applied to a proposition, so that the proof schema might be one of the processes in the proposition schema.

We will also sometimes use the term *process* or *mental process* instead of mental action when we are emphasizing its internal (to the subject) nature. Finally the term *object* will refer to a mental or physical object (avoiding any discussion of the nature of the distinction).

One of our goals in elaborating the general theory is to isolate small portions of this complex structure and give explicit descriptions of possible relations between schemas. When this is done for a particular concept, we call it a *genetic decomposition* of the concept. In this paper, we present only current versions of our genetic decompositions as opposed to showing in detail how they evolved through our investigations and how we used the general theory, observations of students and our knowledge of the mathematics involved to obtain them. Readers interested in fuller discussions can consult the papers listed in the references. We should also point out that although we only give, for each concept, a single genetic decomposition, we are not claiming that this is *the* genetic decomposition, valid for all students. Rather it represents one reasonable way that many students might use to construct a concept.

It is not easy to separate a description of mathematical knowledge from its construction. As Piaget put it, "...the problem of knowledge, the so-called epistemological problem, cannot be considered separately from the problem of the development of intelligence" [27, p. 166]. It is not possible to observe directly any of a subject's schemas or their objects and processes. We can only infer them from our observations of individuals who may or may not bring them to bear on problems — situations in which the subject is seeking a solution or trying to understand a phenomenon. But these very acts of recognizing and solving problems, of asking new questions and creating new problems are the means (in our opinion, essentially the *only* means) by which a subject constructs new mathematical knowledge.

This is where reflective abstraction comes in. Thus, although we might say that mathematical knowledge consists of a collection of schemas, we have little to say about how that knowledge exists inside a person. It does not seem to reside in memory or in a physiological configuration. All we can say is that a subject will have a propensity for responding to certain kinds of problems in a relatively (but far from totally) consistent way which we can (as far as our theory has been developed) describe in terms of schemas. When the subject is successful, we say that the problem has been *assimilated* by the schema. When the subject is not successful then, in favorable conditions, her or his existing schemas may be *accommodated* to handle the new phenomenon. This is the constructive aspect of reflective abstraction to which we referred as forming the main object of our concern.

In this sense, the study of reflective abstraction is complementary to investigation of notions such as *epistemological obstacles* as studied by B. Cornu [4], and A. Sierpiñska [30, 31] or the conflict between *concept image* and *concept definition* as investigated by T. Dreyfus, D. Tall and S. Vinner ([8, 32, 34, 35, 40]). One can think of reflective abstraction as trying to tell us what needs to happen whereas the other notions attempt to explain why it does not. It is possible that our idea of using computer experiences [1,9,10,11] to help students make reflective abstractions can be a way of dealing with these obstacles and conflicts. But that is a matter for another investigation and another paper.

Constructions in advanced mathematical concepts

In the previous section, we isolated five kinds of construction that Piaget found in the development of children's logical thinking: interiorization, coordination, encapsulation, generalization, and reversal. We will reconsider each of them in the context of advanced mathematical thinking to describe how new objects, processes and schemas can be constructed out of existing ones.

Some of the following examples will apply to a single one of the five kinds of construction and others will apply to a combination of two or more of them. Some of the statements we make are based on observations of students and others are only suppositions, derived as a preliminary to observations, from the general theory and our knowledge of the mathematics.

As we make these statements about constructions that we have seen students appear to make or need to make, or as we conjecture that certain concepts could be constructed in these ways, the reader should be aware that we are not suggesting that it is automatic, natural, or easy for students to take these steps. An important aspect of the whole problem of education that we do not consider in this paper is to explain why students do or do not make these particular constructions and what can be done to help them. This is an important issue for research in mathematics education.

An important part of understanding a function that we have observed (unpublished) is to construct a process. For individual examples this means that the subject responds to a situation in which a function may appear (via formula, as an algorithm, or represented by data) and for which there is a process by which the value of the function, for a particular value in the domain, is obtained. Given such a situation, the subject may respond by constructing, in her or his mind, a mental process relating to the function's process. This is a prime example of interiorization.

What is perhaps essentially the same sort of example of interiorization that could occur in a different context is the construction of the notion of linear independence of a set of vectors. One way of understanding this concept might be for the subject to interiorize the act of forming finite linear combinations of the vectors and checking the result, imagine repeating this act exhaustively, and determine that, with one exception, the result is never zero. Once this process is present in the subject's mind, it would seem to be possible that he or she could imagine attaining zero with non-trivial coefficients, resulting in linear dependence. Unless the subject is considering such mental processes in a very active way, it seems to us that the formal definitions may not be much more than meaningless strings of symbols for her or him.

An example of the same kind of mental activity in a completely different mathematical situation could arise in understanding proofs. When the mathematician exclaims (as which of us has not?) that "I can understand each step of the proof, but I don't see the whole picture", it could be the case that he or she is expressing the necessity of interiorizing a whole collection of processes and coordinating them to obtain a single process. The interiorization of the total process can be, in our opinion, the final step in "making a proof your own".

Interiorization may not always be difficult. Most students seem to have little trouble with constructing a mental process for multiplying a matrix and a vector, or two matrices. This could be because there is a straightforward "hand-waving" action, used by most teachers, that is a physical representation of the multiplication and could form an intermediary between the external action and its interiorization. Mathematics becomes difficult for students when it concerns topics for which there do not exist simple physical or visual representations. One way in which the use of computers can be helpful is to provide concrete representations for many important mathematical objects and processes (see Chapter 13).

Turning now to coordination, one of the most important examples that we have seen occurs in the formation of the composition of two functions. Actually, much more is involved here [1,36]. Based on this research, we would like to propose the following psychological description. Composition is a binary operation which means that it acts on two objects to form a third. Thus, it is necessary to begin with two functions, considered as objects. The subject must "unpack" these objects, reflect on the corresponding processes, and interiorize them. Then the two processes can be coordinated to form a new process that can then be encapsulated into an object which is the function that results from the composition. This is much more complicated than simple substitution and perhaps explains why students have so much difficulty with ideas like the chain rule for differentiation, in which it would be necessary to coordinate this view of composition with the notion of derivative. It could also explain those results of [1] in which students seem to improve their understanding of composition as a result of performing computer tasks designed to foster these mental operations.

A whole class of examples of that could be described as coordination of schemas in advanced mathematics is given by the "mixed" structures: topological vector spaces, differentiable manifolds, homotopy groups, etc.

Returning to the realm of high school mathematics, we have seen, amongst students and occasionally even some of their teachers, a fairly shallow understanding of the concept of variable. It seems, for them, to be little more than a letter (as opposed to a number) which can be manipulated according to certain rules that must be memorized, and may stand for a single (possibly unknown) number. From the point of view of our theory, the idea of variable could be more richly explained as an encapsulation (and this is where the object comes from) of the process of substituting (possibly infinitely) many numerical values for the variable (and this is where the manipulations come from — ordinary arithmetic).

Moving a little higher up the age ladder, it is often possible to observe students having difficulty with determining the cardinality of sets such as

$$\{4, \{-3, 2, -1/7\}, \{17, 5\}\}$$

Very often, even undergraduates will think that this set has 6 elements (rather than 3). We suggest that the difficulty is that the students have not encapsulated the sets $\{-3, 2, -1/7\}, \{17, 5\}$ into objects so as to understand the nested structure of the given set.

The indefinite integral forms an important example that can be interpreted as encapsulation together with interiorization. Estimating the area under a curve with sums and passing to a limit is, of course, a process. Students who seem to understand this often have difficulty with the next step of varying, say, the upper limit of the integral to obtain a function. What is lacking, we suggest, is the encapsulation of the entire area process into an object which could then vary as one of its parameters vary. This would then form a "higher-level" process which specifies the function given by the indefinite integral. The complexity of this total process might then help explain why students have such difficulty with not only the Fundemental Theorem of Calculus, but such powerful definitions as

$$\log x = \int_1^x \frac{dt}{t}, \quad x > 0$$

A rather pervasive example that can be interpreted as encapsulation in mathematics is duality. The dual of a vector space, for example, is obtained by considering all of the linear transformations from the space to its scalar field as objects, collecting them in a set, and introducing a natural algebraic structure on this set. It seems to us that this is an act of encapsulation that is esential in this branch of mathematics.

The simplest and most familiar form of reflective abstraction is generalization. According to our investigations, we can say that a subject's function schema, in which functions transform numbers, is generalized to include functions which transform other kinds of objects (once they have been encapsulated) such as vectors, sets, propositions, or other functions. Similary it would seem that the schema of factorization of positive integers can be generalized in this way to factoring polynomials, and then to an arbitrary Euclidean ring. Vectors of dimension two and three can be generalized to include higher, and even infinite, dimensional vectors. All of these and a host of other examples in mathematics seem to involve the application of an existing schema, essentially unchanged, to new objects (which are often the result of encapsulation).

Finally there is reversal of a process. We can mention a number of familiar activities in mathematics that appear to involve the reversal of a process: subtraction and division, solving an equation, inverting a function, logarithms, proving an inequality (in which one often starts with the conclusion, manipulates until something known to be true is obtained, and then sees if the argument can be reversed), and the mysterious choice of expressions such as $\frac{\epsilon}{2\sqrt{M}}$ in proving limit theorems.

In recent experiments (unpublished) we have seen students having a greater difficulty with the concept of "one-to-one" than they have with "onto". Our theory explains this by pointing out that one can think about "onto" directly in terms of the process of a given function. One applies it to everything in the domain, and sees if everything in the range is covered. For "one-to-one" however, one must look at elements of the range and see how many elements of the domain they "come from". This is a reversal of the function process.

Another observation that can be explained in terms of reversal is the comparative difficulty that students have with two problems in composition of functions concerning the relation $H = F \circ G$. One problem is to solve for F given G, H, and the other is to solve for G, given F, H. We have found that the second is harder for students than the first and this can be explained in terms of an additional reversal that is required for the second. Some details are given below on p. 24.

The organization of schemas

In the previous section, we suggested how the construction of various concepts of advanced mathematics could be described in terms of the five forms of construction in reflective abstraction: interiorization, coordination, encapsulation, generalization, and reversal. We offer the conjecture that the construction of all mathematical concepts can be described in these terms. It may be that one or two additional forms of reflective abstraction will have to be added as additional concepts are considered, but we suggest that the five given here tell something like the full story.

Of those concepts (mathematical induction and predicate calculus) for which we have made a more or less complete genetic decomposition ([9,12,13]), our analysis has been greatly influenced by data obtained from observations (interviews, written tasks, computer work, etc.) of students while they are trying to understand the concept in question. The genetic decomposition is then derived from a synthesis of these empirical results, our general theory, and our mathematical knowledge of the concept in question. This is why it takes a long time and has only been done extensively for two concepts. Work on other concepts (e.g., function, limit) is proceeding slowly and, we hope, deliberately.

The following description of the organization of a schema is just a summary of what we have seen in the concepts investigated thus far and, therefore, is somewhat tentative. We give it here in general terms and then, in the next section, see how it looks in the context of mathematical induction and predicate calculus. In addition, with more anticipation than certainty, we will suggest how it might look for the concept of function, after considerably more data has been gathered.

The structure of a schema is displayed in Figure 1.

PLACE FIGURE 1 ABOUT HERE

As we have already indicated, one should not think of a schema statically, but rather as a dynamic activity (or propensity for such activity) by the subject. Moreover, the existence of a schema is inseparable from its continuous construction and reconstruction. Thus, in describing the system in Figure 1 we will try to do several things simultaneously: describe what is there, describe what happens, describe how things are constructed, and refer to some of the examples we have discussed previously. An additional complication is that, as indicated in the picture, a schema is not a linear list of items but rather a circular feedback system. Our description, necessarily linear, must break in at some point. In any case, the following discussion is an alternative way of organizing the five kinds of construction analyzed in the previous two sections. Here we include with them the results of the constructions (objects and processes).

We begin with objects. These encompass the full range of mathematical objects: numbers, variables, functions, topological spaces, topologies, groups, vectors, vector spaces, etc., each of which must be constructed by an individual at some point in her or his mathematical development.

At any point in time there are a number of actions that a subject can use for calculating with these objects. These actions go far beyond the numerical calculation that results in numerical answers. The computation of the homotopy group of a topological space is a calculation. So is the determination of the (topological) dual of a (locally convex topological) vector space. We will return to this example a few paragraphs below when we discuss coordination.

It is possible for a subject to work with actions in ways other than just applying them to objects. First, an action must be interiorized. As we have said, this means that some internal construction is made relating to the action. An interiorized action is a process. Interiorization permits one to be conscious of an action, to reflect on it and to combine it with other actions. For example, the computation of the dual of a particular vector space is an *action* on that object. The idea, independent of any particular vector space, that it may have a dual and it can often be computed, is the *process* that results from interiorizing this action.

Interiorizing actions is one way of constructing processes. Another way is to work with existing processes to form new ones. This can be done, for example, by reversal. A calculus student may have interiorized the action of taking the derivative of a function and may be able to do this successfully with a large number of examples, using various techniques that are often taught and occasionally learned in calculus courses. If the process is interiorized, the student might be able to reverse it to solve problems in which a function is given and it is desired to find a function whose derivative is the original function. This is anti-differentiation or integration, and it too, is first an action and then must be interiorized to become a process. Encapsulating both the differentiation and integration processes — at least to the point of having them as objects of reflection — would seem to be an essential prerequisite for understanding the fundamental theorem of calculus.

Another way of making new processes out of old ones is to compose or coordinate two or more processes. For example, let us return to the dual of an infinite dimensional vector space and imagine (this is pureley conjectural) how a subject might think about it. A subject may have a schema (discussed in the previous section) for constructing the dual of a finite dimensional vector space. If an *infinite* dimensional vector space comes along, then it seems that exactly the same schema can be used to construct its dual, as well. We would say that the new phenomenon (infinite dimensional vector space) has been assimilated to this schema. As mathematical experience goes further, however, this result would not be very satisfactory, and it is particularly convenient to make use of topological structures. If there is, in the subject's schemas, a process for equipping a set with a topology, then this could be coordinated with the vector space schema to obtain a topological vector space. Now within a schema for topological space there should be a schema for the concept of continuous function and within a schema of vector space there should be a notion of linear function. Coordinating continuity and linearity, one can obtain the idea of a continuous linear function. This coordination would permit the subject to extend and reorganize the process for constructing the dual of a vector space to apply to the set of those functions from the original set to the scalar field which are continuous as well as linear, thereby obtaining the topological dual. In such a situation we would say that the schema for duals has been accommodated to the new phenomena (involving topologies) and experiences which made the old schema less than satisfactory.

In addition to using processes to construct new processes, it is also possible to reflect on a process and convert it into an object. Anytime a *set* of functions is considered, it seems necessary to think of the functions in question as objects. Initially, functions are processes and so the subject must have performed an encapsulation in order to consider them as objects. It is important, for example in composition of functions, for the subject to alternate between thinking about the same mathematical entity as a process and as an object. (cf. p. 12.)

A more advanced, and yet more fundamental example where encapsulation may occur is in the concept of a topology. Initially, there is the notion of nearness or convergence, which is a process. One of the accomplishments of twentieth century mathematics is to capture this idea with the device of a collection of subsets (so-called "open sets") which must satisfy certain conditions but is otherwise arbitrary. The interaction (really another form of coordination) between, on the one hand, a collection of sets which may be taken as arbitrary in order to investigate general topological properties related to but not identical with notions of "nearness", and on the other hand, a very specific choice of this collection so as to apply those properties to important concrete situations, say in analysis, and the use of the resulting observations to stimulate the development of further general properties, and so on, has led to a great deal of important new mathematics of both abstract and concrete natures. A key step in this progress may be described as the encapsulation of the process "nearness" to the object "topology".

We conclude this section with a recapitulation of our description of the construction of schemas in the context of the example, already mentioned on several occasions above, of the (topological) dual of a (topological) vector space. This suggestion of a genetic decomposition for the concept of dual is totally speculative in the sense that it depends entirely on our theory and our understanding of the relevant mathematics. We have gathered no data (other than introspection on our own experience) to support our suggestions. On the other hand, it may be interesting for those with a background in mathematics to see that our theory at least *appears* to be reasonably compatible with a topic from the arena of mathematical research. It is an important point that the same ideas that described the thinking of young children and adolescents can be used to talk about higher mathematics.

In the beginning, there are vectors which are the objects and actions on vectors including addition, scalar multiplication and the gathering together of vectors in a set with these operations, to form a *vector space*. This is a schema that we assume that the subject possesses. We also assume that the subject has a schema for functions that transform numbers into other numbers.

The first step, according to our conjecture, is to generalize the function schema to include as a function any process that transforms vectors into scalars. This could then be coordinated with the addition of vectors and their multiplication by scalars by restricting the functions to processes that transform vectors into scalars, but preserve the algebraic operations of addition and scalar multiplication.

We would then say that these functions are encapsulated into objects called *linear functionals* and collected together in a single set. At this point we would like to suggest that, although the assigning of a name like linear functional to a process is closely connected with its encapsulation into an object, it is the encapsulation that is fundamental and gives "meaning" to the name. To name processes without encapsulating them is the essence of jargon. When there is a complaint that a particular discourse has too much terminology and not enough meaning, we feel that the real difficulty is that labels are being assigned without an opportunity for encapsulating that which is being labeled.

In any case, the set of linear functionals can be assimilated to the vector space schema (which may have to be accommodated to this purpose — that is, it may be necessary to project and reconstruct this schema on the higher level of a vector space whose elements are linear functionals) by defining addition and scalar multiplication of these functionals. This can be done very naturally, interpreting the functions as processes and using "pointwise operations". In this way, the set of linear functionals becomes a vector space, called the algebraic dual.

Now comes a major interiorization. What we have been describing is an action applied to a vector space E that constructs its algebraic dual E^*. When this has been interiorized, one has constructed the beginning of duality theory. One can reverse the process to look for a "pre-dual", that is, given a vector space F, can one find a vector space E whose algebraic dual is F? (The answer is yes if E is "finite-dimensional", but otherwise it may or may not be possible.) Or one can perform the process twice. When two instantiations are coordinated, one obtains the bidual E^{**}. The concept of reflexivity (fairly simple in the case of the algebraic dual) has to do with whether $E = E^{**}$.

Next, as we mentioned above, topology and algebra can be coordinated to obtain the concept of topological vector space and the schema for dual can be projected onto this higher plane and reconstructed by introducing considerations of continuity, to obtain the topological dual E' of a topological vector space E.

Again the action of constructing the topological dual can be interiorized into a process and the concepts of pre-dual and reflexivity (much more interesting in the topological case) can be reconstructed and their properties investigated. Even more interesting, the *content* of forming the topological dual can be removed from the *form* of this process (by reflecting on it) and this would give rise to the idea of dual pairs $\langle E, F\rangle$ in which algebra and topology are mixed in free and varying combinations to obtain the modern theory of dual systems in linear topological spaces.

3 Genetic decompositions of three schemas

We will consider three schemas in some detail: mathematical induction, predicate calculus, and function. Our goal is to show how the general theory elaborated in the previous section can be used in possible description of the nature and construction of these specific schemas. Thus in each case we will point out the relevant objects and processes as well as the instances of reflective abstractions that seem to us can be used in constructing them.

The details that we are about to present come from our three sources. First, there is the psychological data that we have gathered through observations of students in the midst of trying to learn these concepts. These experiments are described in full detail in [9,10,11,12,13]. This data, along with the ideas of Piaget formed the basis for the derivation of our theory, which is the second source of the genetic decompositions. That is, for each phenomenon that was observed, we tried to use our theory to describe it, adjusting the theory when necessary. (As the necessity for adjustment occurs less often, our confidence in the theory increases.) The third source of the descriptions is our mathematical understanding of the concepts in question. It seems important that a genetic decomposition should make sense from a mathematical point of view, although it might not be exactly how the mathematician might have analyzed the subject in thinking about how to teach it.

These three sources actually only apply in full to the first two examples: mathematical induction and predicate calculus. Because our data, and the analysis that leads to our conclusions, already appears in the above references, we do not repeat it here. In the case of function, we have gathered only a little data, and because it has not all appeared elsewhere as yet, we make some mention of it, although very limited. Thus the genetic decomposition of function is based mainly on the theory and our mathematical understanding of function. As such, it must be taken as speculation that may form a bridge for future work. As we obtain and analyze data on students' learning the concept of function, it will be interesting to see how close the genetic decomposition postulated here comes to what is derived when the genetic data are taken into account. In a sense, this can provide an indication of the predictive value of our theory as it has been developed so far.

We also wish to repeat the caution that there is no reason to expect a genetic decomposition to be unique, even in a single subject. All we are looking for is one possible scheme or collection of schemes that appears to mathematicians as consistent with their understanding, seems to explain the observations that have been made, and is promising as a tool in designing instruction because it is one possible way that some students might construct the concept.

3.1 Mathematical induction

The aspect of induction that we are interested in has to do with a subject's understanding of the induction process, why it "works" to establish something and how to construct an inducion proof. Ultimately, this has to be coordinated with a notion of infinity but it may be that understanding the induction process is a precursor to constructing a notion of infinity. It would be an interesting investigation to apply, to the concept of infinity, our method of helping students learn induction [9,10].

In the first instance, mathematical induction is a process in that one interiorizes the actions of moving along (as "n increases") from one proposition to the next and, after an initial independent determination, dtermine the truth of a statement by applying a tool (truth of an implication) that was previously constructed.

Mathematical induction is also an object in the subject's general schema for proofs. This means that the induction process must have been encapsulated in order that the subject can reflect on it, along with other methods, when confronted with a theorem to prove, so as to select induction as the method for a particular problem.

The method itself is constructed by working with two major schemas: function and logic. The developments of these two schemas are interwined through various coordinations. We can illustrate the process with a chart as shown in Figure 2.

PLACE FIGURE 2 ABOUT HERE

We start with the assumption that the subject possesses a function schema and a logic schema that are already developed to the point where, for example, the function schema includes the ability to construct a process relating to a particular transformation of numbers (see Section 3.3), and the logic schema can construct statements in the first order propositional calculus (see Section 3.2). In particular we assume that the function schema includes the process of evaluation of a function for a given value in its domain and that the logic schema includes a process for logical necessity, that is, in certain situations, the subject will understand that if A is true then *of necessity* B will be true. Of course we are not asserting that the subject will necessarily be aware of these schemas in this terminology. What we mean, for example, is that the subject will be able to think in terms of plugging a value of a positive integer into a statement and asking if the result is a true statement. This is a function and we can infer from a subject's actions that it may exist in her or his mind as a schema — but we would hardly require young subjects to be aware of it as such in order to understand induction.

The formation of first order propositions is a process in the logic schema which can come from interiorizing actions (conjunctions, disjunctions, implications, negations) on declarative statements (objects). The subject can perform a reflective abstraction on this process to obtain new objects which are the propositions of the first order propositional calculus, on which the same actions can be performed. Consider for example, a simple proposition such as,

$$(P \vee Q) \wedge R$$

where P, Q, and R are simple declarations. The formation of the disjunction $P \vee Q$ can be described as an action on the statements P, Q. It is not just the action of putting these symbols in this expression. The subject must also construct a mental image involving the two statements and the determination of the truth or falsity of the disjunction in various situations. If nothing further is done after this action is interiorized, then it will not be possible to combine this with R to get the full proposition. First, the disjunction process must be encapsulated to form a new object $(P \vee Q)$ which is a statement that can be conjoined with another statement, such as R. Note how the use of parentheses in mathematical notation corresponds to encapsulation.

Iterating this procedure, the subject enriches her or his logic schema to obtain a host of new objects consisting of first order propositions of arbitrary complexity. Next the function schema comes in. We are assuming that this schema can be used by the subject to construct processes that transform numbers (for example an integer) into other numbers. It must be generalized to permit the subject to construct processes that transform positive integers into propositions, to obtain what we shall call a *proposition valued function of the positive integers.* Consider for example, a statement such as,

Given a number of dollars, it is possible to represent it with \$3 chips and \$5 chips.

For such a statement, the subject must construct a process whereby, for each positive integer n, a proposition is constructed which is the same statement, but with " a number of dollars" replaced by that value of n. This is the proposition valued function. In order to evaluate it, the subject must construct another process whereby, given n, a search is made and it is determined whether it is possible to find non-negative integers k, j such that

$$n = 3j + 5k$$

It is useful for the subject to discover that the *value of this function* is *true* for $n = 3, 5$, *false* for $n = 1, 2, 4, 6, 7$ and then appears to be *true* for higher values.

It is only at this point that the subject can realize that the problem of "proving" the statement consists of determining that the value of the function is *true* for *all* values of $n \geq 8$. For this, the proof schema can be invoked. If it contains the schema for induction, it can be used, if not, further (re-)construction must take place. In describing this construction, we reiterate the point that, in the context of this theory, it is never clear (nor can it be) whether one is talking about a schema that is present or a schema that is being (re-)constructed.

Before going on with the description, there is a side issue that should be considered. Whether the subject is able to construct a proposition valued function of the positive integers to deal with a particular statement depends not only on the existence of the schemas we are talking about, but also may require additional knowledge about the particular situation — so-called "domain knowledge". Thus, although the above example of chips is probably well within the domain knowledge of most students who find themselves trying to learn induction, others may not be. We have found, for example, that the following statement provides difficulty for university undergraduates.

An integer consisting of 3^n identical digits is divisible by 3^n.

The trouble could lie in understanding the relationship between the value of an integer and its representation with digits. It is a sort of "grown-up" version of the difficulty with the concept of place value and it suggests that many students have not really constructed this concept — at least in a sufficiently powerful form.

Returning now to the construction of proof by induction, the next development provides an example of a cognitive step which our research has pointed out as providing a serious difficulty, whereas if one takes only the mathematical point of view, there is not even a step that needs to be taken. This is the case even though it relates to an overt difficulty encountered by everyone who has tried to teach mathematical induction.

We are referring to the notion that the essential point in an induction proof is that one does not prove the original statement directly, but rather the implication between two statements derived from it. This is the major difficulty for students. It requires a cognitive step which is not necessary as a mathematical step. To explain, let us denote by P the proposition valued function to be proved. Now $P(n)$ can be any proposition, in particular, it can be an implication. Therefore, if we define the proposition valued function Q by

$$Q(n) = (P(n) \Longrightarrow P(n+1))$$

then, *from a mathematical point of view* there is nothing new in Q, that is, once one understands P then, as a special case, one understands Q. We have observed, however, that with students, this is not the case from the *cognitive point of view*. In the first place, implications are the most difficult propositions for students and they are generally the last to be encapsulated. Furthermore, there is a difference between constructing P from a given statement and constructing Q *from* P. This is the step which must be taken. If there is some subtlety here, then it might help explain the difficulty that students have at precisely this point.

To summarize, this step appears to require the encapsulation of the process of implication so that an implication is an object and can be in the range of a function, the generalization of the function schema to include implication valued functions, and the interiorization of the process of going from a proposition valued function of the positive integers to its corresponding implication valued function.

The next step is to add to the logic schema a process which we shall call *modus ponens*. This process is an interiorization of an action applied to implications (assuming as above that they have been encapsulated into objects). The action consists of beginning at the hypothesis, determining that it is true, and then "crossing the bridge" to the conclusion and asserting its truth.

Finally, there is a coordination of the function schema, as it applies to an implication valued function Q (obtained from a proposition valued function P) and the logic schema as it applies to the process modus ponens which has just been constructed. Included in the function schema is the process of evaluation, that is, sampling values n of the domain (positive integers in this case) and computing the value of the function, $Q(n)$, that is, $P(n) \Longrightarrow P(n+1)$. Suppose that it has been extablished that Q has the constant value true. The first step in this new process which must be constructed is to evaluate P at 1 and to determine that $P(1)$ is true (or, more generally, to find a value n_0 such that $P(n_0)$ is true). Next, the function Q is evaluated at 1 to obtain $P(1) \Longrightarrow P(2)$. Applying modus ponens and the fact (just established) that $P(1)$ is true yields the assertion $P(2)$. The evaluation process is again applied to Q but this time with $n = 2$ to obtain $P(2) \Longrightarrow P(3)$. Modus ponens again gives the assertion $P(3)$. This is repeated ad infinitum, alternating the processes of modus ponens and evaluation. Thus we have a rather complex coordination of two processes that we believe leads to an *infinite* process.

This infinite process is encapsulated and added to the proof schema as a new object, proof by induction.

3.2 Predicate Calculus

The predicate calculus schema appears to be obtained through a reconstruction of a schema resulting from coordinating a schema for first order propositional calculus with a function schema. The construction is illustrated in Figure 3. According to this analysis, the objects in the propositional calculus schema are the propositions. The most important process is the determination of the truth or falsity of a proposition. Other processes include the formation of new propositions by the standard logical operations such as conjunction, disjunction, implication and negation. They also include the process of expressing an English statement in the formal language of symbolic logic and translating from that syntax back to English. Then of course there are all the usual tasks that students are asked to perform such as manipulation of the formulas, construction of truth tables, determination of the validity of arguments and so on. Finally, we can mention the process of reasoning about a statement, for example, to know if the truth or falsity of the statement

$$(P \Longrightarrow Q) \vee (\mathrm{not}(\mathrm{Q} \wedge \mathrm{R}))$$

is determined once you know that $P \Longrightarrow R$ is false.

PLACE FIGURE 3 ABOUT HERE

Amongst the various manipulations of logical expressions, one in particular will be important in the sequel. That is the process of applying the conjunction operation ("and" or $\wedge$) to a set of propositions as in

$$(x_1 > b_1) \wedge (x_2 > b_2) \wedge \cdots \wedge (x_n > b_n).$$

There is a similar process for disjunction ("or" or $\vee$). This is a manipulation of symbols, but there is an underlyinmg process connected with the truth value of the resulting proposition.

In a sense, the objects in the first order propositional calculus are constants. In an expression such as $(P \Longrightarrow Q) \vee (\mathrm{not}(\mathrm{Q} \wedge \mathrm{R}))$ the quantities P, Q and R are constants whose value may be unknown, but fixed. The subject's thinking about such matters can be elevated to a higher plane when the propositional calculus schema is coordinated with the function schema (appropriately reconstructed on this higher plane) to consider such an expression as determining a function — in this case of the three variables, P, Q and R. This is the beginning of the predicate calculus schema. Of course, a part of this coordination and reconstruction was discussed already in the previous section for the special case of proposition valued function of the positive integers.

As before, an important new process that can be constructed is the iteration (in the subject's mind) through the domain of a proposition valued function checking the truth or falsity of the resulting proposition for each value of the variable. Consider, for example a statement such as

Given a car in the parking lot, if the tire fits the car, then the car is red.

Here, *tire* may be considered to be a constant, but *car* should be thought of as a variable whose domain is the set of cars in the parking lot. There is an obvious action of walking through the parking lot, checking each car to see if the tire fits and, if it does, seeing if the car is red. When such a statement appears in a mathematical context, as in

$$\text{Given } x \in \mathrm{domain}(F), \text{ if } |x - x_0| \leq \delta, \text{ then } |F(x) - F(x_0)| \leq \epsilon$$

then the mental process seems to consist in looking at each $x \in \text{domain}(F)$ to see if $|x - x_0| \leq \delta$ and, if so, seeing if $|F(x) - F(x_0)| \leq \epsilon$.

This iteration process must now be coordinated with one of the two processes we mentioned earlier: applying conjunction or disjunction to a set of propositions. The resulting process can be encapsulated which leads to a single existential or universal quantification as in

There is a car in the parking lot such that if the tire fits the car, then the car is red.

or

$$\forall x \in \text{domain}(F), \ \ |x - x_0| \leq \delta \Longrightarrow |F(x) - F(x_0)| \leq \epsilon$$

We call this a *single-level quantification.*

The single-level quantification creates new objects which are again propositions so that all of the previous processes of logical operations, negation and reasoning about statements are reconstructed on this higher plane. Particularly important for understanding many mathematics topics is the interiorization of a statement given as a quantification. The subject seems to need a strong mental image of the iteration and quantification process that we have described in order to relate the statement to the mathematical situation that is being considered.

Next comes *two-level quantifications* in which two (usually different type) quantifiers are applied in succession to a proposition valued function of two variables. For example, the statements we have considered may be extended to obtain,

For each tire in the library, there is a car in the parking lot such that if the tire fits the car, then the car is red.

or

$$\exists \delta > 0 \ni \ \forall x \in \text{domain}(F), \ \ |x - x_0| \leq \delta \Longrightarrow |F(x) - F(x_0)| \leq \epsilon$$

The process which we just described for constructing single-level quantifications ended with an encapsulation so that the result becomes a proposition which is a mental object. Note that the effect of a quantification is to eliminate a variable. If the original proposition valued function had two variables, then the resulting object actually depends on the value of the other variable and the schema for single-level quantifications can again be applied to this proposition valued function. For example, in the case of the tires and cars, the existential quantification over cars results in a proposition valued function of the single variable, *tire*. This function can then be subjected to a universal quantification to obtain a single, constant proposition. Thus, when analyzing a statement which requires a two-level quantification over two variables, the subject can begin by parsing it into two quantifications. There is an inner quantification over one of the variables in a proposition valued function of two variables. There is also an outer quantification over the other variable. What we have described is a coordination of these two quantifications to obtain a process which will be a two-level quantification. In order to proceed to higher-level quantifications this new process is again encapsulated to obtain a new object. Once it is encapsulated, it can then be subjected to the same processes (thereby generalized) as were the single level quantifications.

Given a statement which is a three-level quantification, such as the definition of continuity of F at x_0,

$$\forall \epsilon > 0, \exists \delta > 0 \ni \forall x \in \text{domain}(F), \; |x - x_0| \le \delta \Longrightarrow |F(x) - F(x_0)| \le \epsilon,$$

the subject can group the two inner quantifications and apply the two-level schema to again obtain a proposition which depends on the outermost variable (ϵ in this case). This proposition valued function is then quantified as before to obtain a single proposition. The entire procedure can now be repeated indefinitely to obtain quantifications of any level. At each level, the same processes of logical operations, negation, reasoning, etc. are reconstructed.

3.3 Function

As we indicated earlier, our thoughts about the function concept are highly speculative and based mainly on the general theory and our understanding of this concept from the mathematical point of view. Our purpose for including it and giving some examples of preliminary data is to illustrate the explanatory power of our theory and to set guideposts for subsequent empirical work. In the past decade, the function concept has been investigated by a number of authors (see especially [6,6,7,8] in ways that are quite different from the approach described here. For a fuller discussion of research on learning the concept of function, see Chapter 7.

For most students, and indeed for many scientific workers, the idea of function is completely contained in the "formula". If you ask students for an example of a function, you will often get an algebraic expression such as $x^2 + 3$ with no mention of any kind of transformation. Just as with the concept of variable in which the student insists that x "stands for" a *single* number (which may not be known), the concept of function as formula has a very static flavor.

There are a number of ways in which such a function schema is inadequate. For one thing, the objects are restricted to those functions which can be conveniently expressed with a formula. This may suffice for elementary mathematics but it will not do for advanced mathematical thinking. When a function is the same as a formula, the action of evaluation on this object consists of plugging in numbers for letters and composition of two functions is restricted to substitution of a formula for each occurence of a letter. The notions of domain and range have no place in this schema and graphs, while manageable in themselves (because of their concrete and visual nature), have no connection with functions for the student with a function-as-formula schema. When the graph cannot be "seen" (as is the case with the characteristic function of the irrationals), then the student is unable to think about it.

A more powerful schema for functions will involve interiorization of actions. When a subject perceives a situation that can be dealt with in terms of a function, then we suggest that he or she can view the situation as an action on objects that transforms them into other objects. This action is interiorized. Thus, an important part of what it means to know a function is to construct a certain kind of process that can be used to make sense of a certain kind of phenomenon. Some may refer to this as a mental representation of the function, but we prefer to avoid such terminology because of its tendency towards the misleading suggestion that the internal process is a copy of some "external reality". The important point is that when a function is known as an interiorized process, then this knowledge has a dynamic flavor which affects the nature of the subject's interaction with the function situation.

Evaluation becomes the action of taking a particular value (in the domain of the function) and performing the process on it to obtain a new value (in the range of the function). It may then be possible for the subject to coordinate a function's process and its graph. That is, there is the understanding that the height of the graph of a function f at a point x on the horizontal axis is precisely the value $f(x)$. The subject can then relate to the full power of graphing which is the relationship between the physical shape of the graph and the behavior of the function.

Several important ideas in mathematics can be described as doing some of the things we have discussed with the process of a function. For example, the coordination of two processes and the composition of the functions (see [1]). A function's process can be reversed, thereby obtaining the inverse function. It is by reflecting on the totality of a function's process that one makes sense of the notion of a function being *onto*. Reflection on the function's process and the reversal of that process seem to be involved in the idea of a function being *one-to-one*.

As mentioned above, we have done some preliminary empirical work relative to the points in the preceeding paragraph. We find, for example, that students seem to have more difficulty with the concept of one-to-one than with onto. We suggest that the presence of the reversal in one-to-one explains this observation. Similarly on several occasions we have given subjects the following kinds of problems relative to three specific functions, F, G, H. (See [1] for details.)

1. Given F, G find H such that $H = F \circ G$.
2. Given G, H find F such that $H = F \circ G$.
3. Given F, H find G such that $H = F \circ G$.

Of course the first is much easier than the other two, and we find invariably that the third is harder than the second. We can suggest an explanation derived from our theory. The first kind of problem seems to require only the coordination of two processes that, presumably, have been interiorized by the subject. The second, however may require that the following be done for each x in the domain of H.

2a. Determine what H does to x obtaining $H(x)$.
2b. Determine what G does to x obtaining $G(x)$.
2c. Construct a process that will always transform $G(x)$ to $H(x)$.

The third kind of problem may be solved by doing the following for each x in the domain of H.

3a. Determine what H does to x obtaining $H(x)$.
3b. Determine value(s) y having the property that the process of F will transform y to $H(x)$.
3c. Construct a process that will transform any x to such a y.

Comparing 2b. with 3b. (the only point of significant difference), we can see that 2b. is a direct application of the process of G whereas 3b. requires a reversal of the process of F.

It is perhaps interesting to note that this difference in difficulty (between 2 and 3), which is observed empirically and explained epistemologically, is completly absent from a purely *mathematical* analysis of the two problems. They are, from a mathematical point of view, the calculation of $H \circ G^{-1}$ and $F^{-1} \circ H$, respectively, which appear to be problems of identical difficulty. This seems to be another important example in which the psychological and mathematical natures of a problem are not the same (cf. p. 20).

Another situation in which relative difficulty can be explained by the requirement of reversing a process occurs in the development of children's ability in arithmetic. According to Riley, Greeno and Heller [29, p. 157], "Problems represented by sentences where the unknown is either the first ($? + a = b$) or second ($a + ? = c$) number are more difficult than problems represented by equations where the result is the unknown ($a + b = ?$)." Obviously the first two problem types involve a reversal of the process which, in the third type, can be applied directly.

A number of important mathematical activities may require that the function schema be reconstructed at yet a higher level where a function is not only an interiorized process, but as a result of encapsulation, this process can be treated as an object by the subject. One representation that could help with this is the set of ordered pairs (with the "uniqueness to the right" condition) and another is the graph. We refer to Chapter 7 for a discussion of some of the difficulties in this connection. In order for a function to be the result of a mathematical activity (such as solving a differential equation or setting up an indefinite integral) it must be an object. Similarly, it seems to us that the elements of a set must be (epistemological) objects and so that all of functional analysis with its sets and even structured spaces of functions depends on the object nature of a function.

At the same time, and this may be a further reconstruction of the function schema, it seems necessary in many situations that the subject think of a function simultaneously (or at least in rapid succession) as both a process and an object. Consider, for example, the various binary operations on functions such as pointwise addition, pointwise multiplication or composition. In reflecting on the addition of two functions, the subject must see this as a binary operation which takes two objects and transforms them in a new, third object. To actually do this, however, it would seem that the original two objects must be unpacked or "decapsulated" back into processes, these two processes coordinated (by means of "pointwise addition") and the resulting process encapsulated into an object which is the new function that appears as the result of the operation of addition. The same kind of description can be used, as we have indicated above (see p. 12), for composition of functions.

As a final example, consider how complex, in these terms, is the following mathematically straightforward statement.

> In the semigroup $\hom(G, \circ)$ of endomorphisms of a group G under the operation of composition, the subset of those endomorphisms which are isomorphisms form a group.

From our point of view it seems that to understand this statement (and check that it is true) the subject must think of functions as objects since they form a set, and later a subset, and then understand composition as we have described it to get a firm grasp on $\hom(G, \circ)$. Now, in dealing with the group axioms, the cognitive interpretation of function goes back and forth between process and object. The two interpretations must be coordinated in order for the subject to grasp the somewhat subtle idea that the group identity is the identity function and the group inverse of a function is its function theoretic inverse — and this connection is not exactly an accident.

4 Implications for education

We conclude this chapter with some comments on teaching mathematics in light of the theory we have expounded. Our theory does not have anything to say about the affective aspects of the teaching/learning situation. In particular, we have ignored Piaget's notion of equilibration [26] which for him was the driving force behind the (re-)construction of schemas. We have also omitted consideration of various issues such as discovery versus guided learning, and large classes versus individual instruction versus small-group problem solving. The main implication for education that our theory has, as far as we have taken it, is that, whatever happens, in or out of the classroom, the main concern should be with the students' construction of schemas for understanding concepts. Instruction should be dedicated to inducing students to make these constructions and helping them along in the process.

We can offer one general conjecture about motivation. Whatever is the mechanism (*le source* according to Piaget and Garcia [ ]) that moves students to make cognitive constructions, to learn, it seems to us to be a very natural human drive, on a par with the drive for food or sex. We admit that this suggestion is inconsistent with the experience of most mathematics teachers, especially at the post secondary level, where students, other than those with obvious talent for mathematics, do not seem to be interested at all. Our conjecture is that this is due to the overall approach in the traditional clasroom, where the goal, *as presented and defended by the teacher*, is for the student to develop skills in computational procedures, to display these skills on examinations, and to "get a good grade". For reasons which we will elaborate below, the student cannot learn these procedures through understanding, whereas he or she is presented by the teacher with a conflict-free way out — imitate and memorize. Unsurprisingly, most students accept the offer and take this route. But imitation and memorization do not lead to cognitive constructions and the result is that the students' desire to learn through growth is suppressed. He or she is "turned off to mathematics".

Our experience has been that when a student is presented with concepts that he or she is capable of understanding, when the constructions are possible for the student, and if this capability is apparent to the student, then a natural drive to learn, to understand, to construct is released and the level of effort and concentration on mathematical ideas leaves little to be desired. This happens even in the presence of difficulty, when the student is confronted with mathematical problems that her or his existing schemas cannot handle. As long as there is something for the student to think about, as long as he or she perceives that cognitive activity is leading to some sort of growth that could, eventually, lead to a solution of the problem, then there is little difficulty in maintaining the students' interest.

We will present, therefore, some examples of how traditional teaching methods do not relate to conceptual understanding as the theory presented here explains it and close with a few brief words about what directions an alternative approach might take.

Inadequacy of traditional teaching practices

If we are correct in our position that learning consists of applying reflective abstraction to existing schemas in order to construct new schemas for understanding concepts, then it is a trivial but critical observation that a schema can not be constructed in the absence of prerequisite existing schemas. Traditional teaching often ignores this.

Consider, for example, a lecture on induction which begins, "Today, we are going to learn how to make proofs by induction". This statement assumes that the listener has a "proof schema", that is, he or she is conscious of various methods of proof which could be applied in a given situation and is therefore capable of adding a new one. For any students in the class who do not possess such a schema, the statement is not very meaningful. It gets worse when actual problems, theorems to be proved by induction, are introduced. If a student's function schema does not include functions that deal with transforming integers into propositions, then the very statement of a problem can be meaningless. Many students are probably somewhat bemused when, later, the teacher is roaring the admonition, "You don't prove the statement for every n, you prove the implication from n to $n+1$!" If proof is meaningful at all, it means that you prove *something*. For students who have not encapsulated the process of implication and for whom proposition valued functions of the positive integers are not objects, there may be no "somethings" in that admonition. If such prerequisites are not dealt with, then it is no wonder that the student gives up on trying to understand (he or she does not have the right tools) and, because success on examinations is both essential and possible, looks for something to imitate.

Another kind of difficulty arises with the predicate calculus. For many teachers, understanding the meaning of a statement such as,

For every function f in $\mathcal{A}$ there is another function g in $\mathcal{A}$ such that $f(g(x)) = x$ for all x

is essentially a language problem, not very different from understanding statements such as,

Every student in the class has a counselor who will be available to give advice every Monday at 9am.

But there is much more than language present — in both statements. For the first, according to our theory, the student must have constructed (in her or his mind) a set of functions, interiorized a process of iterating through this set picking an object, iterating again to pick another object, and converting the two objects back to their function processes so that it is possible to iterate once again, this time through the domain of the functions, testing the equality. Only after these constructions are made can the problem be treated linguistically. From our point of view, it is the constructions that provide the essential difficulties, the language aspect being fairly trivial. Similar comments can be made about the second statement which most students have little difficulty in understanding. This is because each construction required to understand the second statement is made naturally, in the course of normal student life and every day experience.

This point about languages, if generalized, suggests to us that the traditional lecture itself, depending largely on linguistic transmission, is not very useful in helping students acquire concepts in mathematics. Mental objects and processes, although they may well exist in the mind of the teacher, cannot be transmitted verbally, or even with pictures, to listeners. It is necessary that the listener engage in active construction.

Another difficulty, related to the problems of imitation, memorization, and verbal transmission arises with examples. It is an article of faith with most mathematics instructors that "lots of examples" must be an integral part of any instructional treatment. It is certainly the case that involvement with examples, whether it be doing exercises or thinking about illustrations and demonstrations, serves to reinforce the concepts that are present in the mind of the subject. We suggest, however, that working with examples may not help very much with the *construction* of concepts. Indeed, we agree with Tall [33] and it is a major aspect of our theory that understanding mathematical ideas come from sources other than looking at many examples and "abstracting their common features", which is what happens if there is only empirical abstraction. Something more is needed and we suggest that it is precisely the construction aspects of reflective abstraction that we have discussed. It is not clear that more than a very few examples are necessary to construct a concept; in some cases (such as the integers in the initial construction of the concept of a ring) a single example might suffice to induce appropriate reflective abstraction. As we have said, we cannot in this chapter give full consideration to the question of how to induce conceptual learning, but one might well reflect on the contrast between the repetitive examples that seem to be required by conventional wisdom and the single, representative example which so often seems to be in the mind of the mathematician who understands a particular concept. Tall [33] has referred to this as the *generic example* and it is a promising notion well worth further investigation.

We would go farther in our critical view of repetitive examples and suggest that the practice can even be harmful. Yes, the effect of practice will be to reinforce structures that are present. But we would raise the question, what structures are these? Are they part of a student's concept image which conflicts with the concept definition (see [32])? Consider what happens when a teacher is explaining, with reference to conceptual understanding, how to solve a certain kind of problem. As we have indicated, the student may not be able to understand the concept behind the method. A general investigation of what drives cognitive development may reveal that whenever a subject is subjected to phenomena, *some sort of construction takes place*. To say that the student does not understand could mean that the student has not and does not construct an appropriate schema for the concept being explained. But if it is the case that something *is* constructed, then it would have to be an inappropriate schema. This result is not inconsistent with what teachers seem to observe in their students after making explanations. What, then, will be the effect of following the explanation with "lots of examples". The inescapable conclusion is that the incorrect interpretations will be reinforced, and teachers will pay a heavy price later on in efforts to correct students' misunderstanding. This may well be a source of epistemological obstacles [4].

This argument is not sophistry. It is offered as an explanation of a phenomenon in education that seems to be generally recognized, but not very well understood. It seems that VanLehn was referring to it quite specifically when he wrote, in a study of the procedural "bugs" observed in students doing subtraction, "When a student has just invented a bug, practice may solidify the bug in memory, thus making remediation more difficult" [39, p. 47]. It is possible that this effect also explains the near impossibility of disavowing undergraduates of various misconceptions observed by Tall [32], Cornu [4] and others concerning the concept of limits as well as the persistence, in the face of a variety of instructional treatments, of reversal errors in algebra [3].

It may be argued that these difficulties can be avoided by giving both examples and non-examples with the examples graded so as to display various features gradually. This could be reasonable, but not if the decomposition is based on no more than the curriculum developer's understanding of the mathematics. Also, there is no certainty that the student will see the examples in the same way that the instructor did. Finally, this really avoids the issue which is that in order to construct a mathematical idea it is necessary to be mentally active. The really important issues in mathematics education have to do with the nature of this activity and what can be done to foster it.

We do not conclude from this discussion that practice with examples should be eliminated. In addition to reinforcing concepts, they may be important for students to become facile with calculations, to develop a "feeling" that something is wrong, or that it all "hangs together properly". Indeed, it is pure speculation but it may be that practice with a process will tend to induce the subject to encapsulate it. It could be that this is the essential point in the relationship between procedural knowledge and conceptual knowledge [15]. We do not, therefore reject examples and practice. We only caution the instructor to pay attention to what concepts the students have and what exactly is being reinforced when they are set to do "all the even numbered exercises". It is also important to be aware of the *types* of mistakes that students make, how he or she tries to justify an answer (whether it is "correct" or not) or just explain how it was obtained.

What can be done

At this point we must conclude, not, unfortunately, with a prescription for putting things right, but with a brief indication of a research and development program that we are engaged in with the hope of constructing a viable alternative to traditional practice for helping students develop advanced mathematical thinking. There are important connections between what is written here and the ideas found in [7,36].

Our instructional approach to fostering conceptual thinking in mathematics has four steps.

- Observe students in the process of learning a particular topic or set of topics to see their developing conceptual structures, that is, their concept images.
- Analyze the data and, using these observations, along with the theory we have elaborated in this paper and the designer's understanding of the mathematics involved, develop a genetic decomposition for each topic of concern that represents one possible way in which a subject might construct the concept.
- Design instruction that attempts to move the student along the cognitive steps in the genetic decomposition; develop activities and create situations that will induce students to make the specific reflective abstractions that are called for.
- Repeat the process, revising the genetic decomposition and the instructional treatment, and continue as long as possible or until stabilization occurs (if it does).

To this general description we can add the fact that, in designing instruction, we have found activities with computers to be a major source of student experiences that are very helpful in fostering reflective abstractions. For example, it seems that if a student implements a process on a computer, using software that does not introduce programming distractions (such as complex syntax, constructs that do not relate to mathematical ideas, etc.), then the student will, as a result of the work with computers, tend to interiorize the process. If that same process, once implemented, can be treated on the computer as an object on which operations can be performed, then the student is likely to encapsulate the process. It turns out to be possible to create such opportunities for computer experiences relative to reflective abstractions necessary to construct a wide variety of concepts in mathematics, but that is a topic for another chapter.

We have used this approach to design instruction, with extensive involvement of computers, to help students learn mathematical induction, predicate calculus and many other topics in discrete mathematics. Present efforts are directed towards applying the method to functions and to calculus.

Acknowledgements

We would like to thank P. Davidson, T. Dreyfus, H. Sinclair, L. Steffe, and D. Tall who read and commented on a draft of this chapter.

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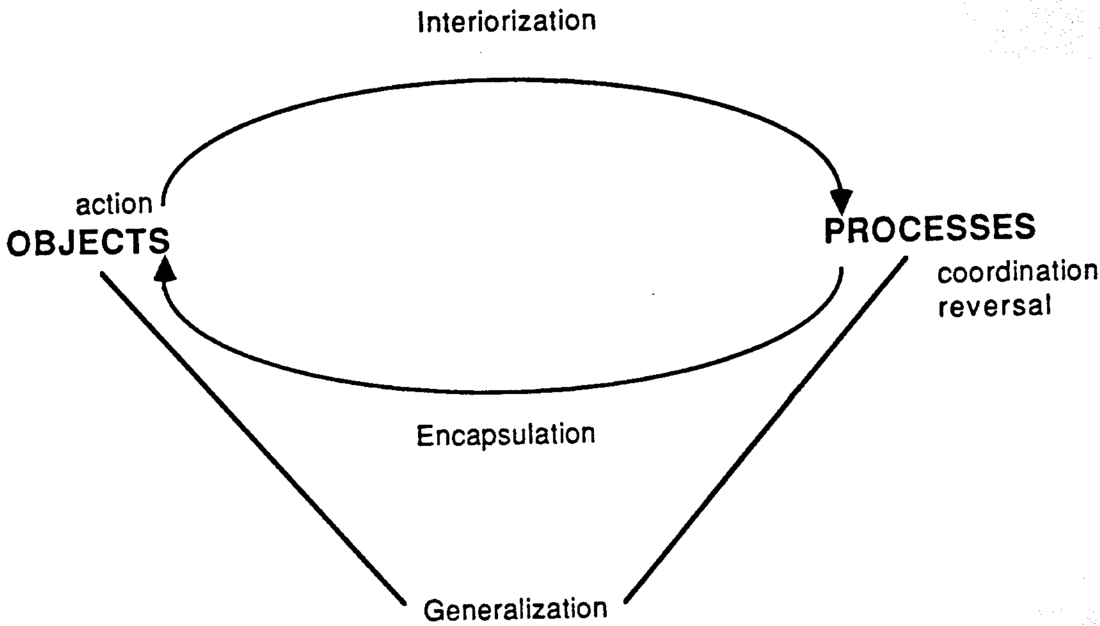

FIGURE 1. Schemas and their Construction

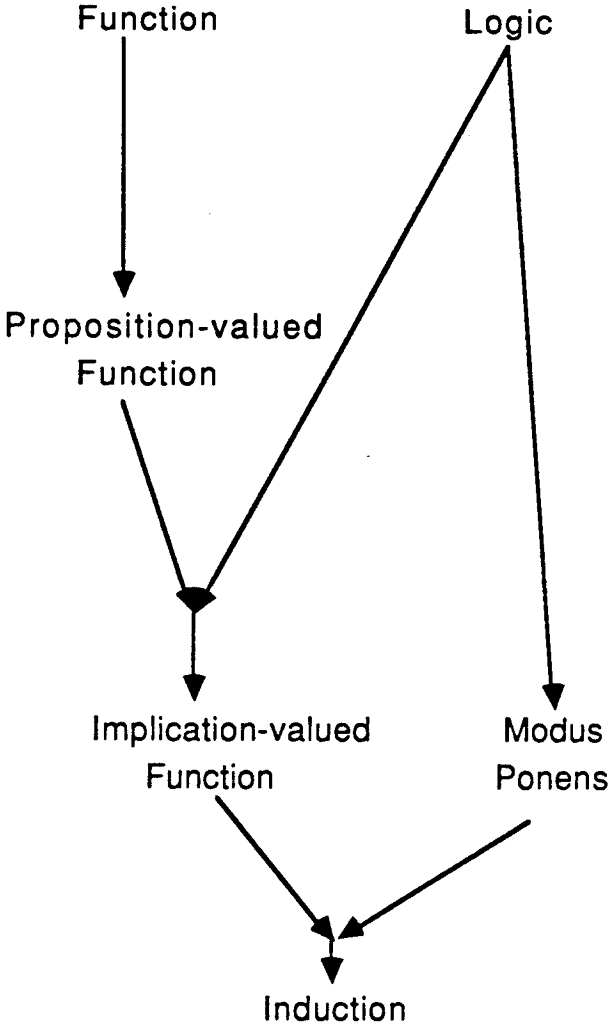

FIGURE 2. Genetic Decomposition of Mathematical Induction

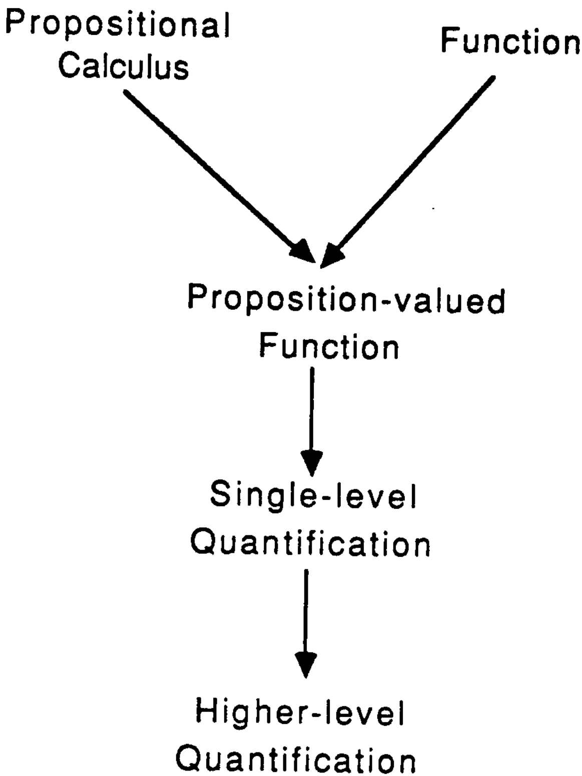

FIGURE 3. Genetic Decomposition of Predicate Calculus