Reaction to James Kaput's Paper

Democratizing Access to Calculus: New Route to Old Roots

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1 The Role of History in Mathematics Education

The history of mathematics in general and calculus in particular has been thoroughly studied over the past few decades. There are, for example, 14 books and papers cited in Kaput's chapter. More recently, a few researchers in mathematics education have taken the point of view that the historical development of a mathematical topic, in particular the difficulties that were encountered and the obstacles that were overcome, can tell us something about how an individual might learn — or fail to learn — that topic. A simple and lyrical way of expressing this is that ontogeny recapitulates philogeny.

Some researchers, (see, for example, Cornu, 1983, Sierpiǹska, 1985), have worked with the notion of epistemological obstacle as a context for making use of the lessons of history. According to these authors, in order to understand a particular idea or set of ideas, it is necessary to develop certain notions at a particular moment and then later, with considerable struggle, replace these notions with more sophisticated versions. Thus, to see a function as nothing more than a single algebraic expression is useful and even necessary at some point, but later, when a more powerful function conception is required, there is a difficulty because the old, familiar, useful idea is not easily dispensed with. This situation is referred to as an *epistemological obstacle*. According to this point of view, the study of historical difficulties and delays in development can provide us with important clues about obstacles that are likely to face our students.

In spite of this interest in the relationship between learning in an individual, or, to use a Piagetian term, *psychogenesis*, and historical development in a society, there exists, as yet, no complete synthesis of these two fields of investigation. For the most part in the literature, a study will be concerned with history and perhaps make an occasional reference to educational implications, or it will take the historical development as given and try to apply what appears to be known to questions of individual learning or psychogenesis. There are not many attempts to study historical development and psychogenesis together, to look at history from a mathematical-epistemological perspective and simultaneously to understand learning in an individual from an historical point of view.

Perhaps the first to have attempted this is Jean Piaget who, with Rolando Garcia

(Piaget & Garcia, 1989), made what Kaput refers to as "perhaps the deepest and most complete analysis of the parallels between historical and individual development" (p. 6[1]). In my opinion, the paper of Kaput is a second, although very different, attempt at a systematic, coordinated study of historical development and psychogenesis.

Before turning to a discussion of Kaput's paper on its own terms, it will be useful to compare the perspective of Piaget and Garcia with that of Kaput. This comparison shows how an interpretation of history can have at least as much to do with the interpreter, as it does with any set of historical "facts" — if indeed such things exist. For example, both studies examine the work of Oresme. Kaput sees Oresme as involved in a "struggle for representations" whereas Piaget and Garcia see him as spending a long time (and not fully succeeding) in making what I would call encapsulations of processes — that is, the conversion of a process to a mental object to which actions can be applied. The point I wish to make is that the difference between the two interpretations is not due to anyone getting their facts wrong, or seeing references missed by the other(s). The difference lies in the point of view, or theoretical perspective on which the intepretation of the facts is based. This observation is important in its own terms, but it also has pedagogical implications: different interpretations of the history and different theoretical perspectives lead to different conclusions about pedagogy. I will return to this theme in the section entitled "The relation between history and curriculum design".

1.1 The perspective of Piaget and Garcia

It is important to note that Piaget and Garcia are *not* mainly concerned with establishing correspondences between historical development and psychogenesis with respect to the content of mathematics and science. In addition, there is nothing in their book that explicitly addresses any pedagogical implications of these ideas. They do, however examine the parallels between the mechanisms of historical evolution of certain major ideas and the mechanisms of development of concepts within individuals.

It is true that in the case of the evolution of physics from Aristotle to just before Newton they establish a "very direct" correspondence between four historical periods and four stages in psychological development (Piaget & Garcia, p. 26). They feel, however, that "...it would evidently be absurd to generalize this type of parallelism of contents in the case of scientific theories proper, such as those which emerged between Newton's mechanics and Einstein's relativity" (op. cit. p. 27). Similarly, they point out that "The historical evolution of geometry ... goes far beyond anything that can be observed in the elementary stages" (op. cit. p. 111).

They are quite explicit about their goal being "...not to set up correspondences between historical and psychogenetic sequences in terms of content, but rather to show that the mechanisms mediating transitions from one historical period to the next are analogous to those mediating the transition from one psychogenetic stage to the next" (op. cit. p. 28).

[1] Page numbers without further identification refer to the Kaput chapter.

Indeed, there is a sense in which this book, which was Piaget's last publication, is a culmination and summing up of his life work. In a very brief (op. cit. pp. 26-29) listing of the mechanisms common to historical development and psychogenesis we find all the main pillars of Piaget's work in the last decades of his life: reflective and empirical abstraction, the inferential aspect of the relation between observing subject and observed object, the synthesis of differentiation and integration, causality, stages of development whose order is fixed, and, what is for Piaget perhaps the most important because of its relation to biological organization (op. cit. p. 274), the intra-, inter-, and trans- levels of thought, always appearing in that order with transition driven by equilibration and implemented via reflective abstraction. The book attempts to show that these mechanisms of change account for both historical development of ideas and the development of concepts in an individual. Thus, interpretations of history appear as support for Piaget's general theory of genetic epistemology.

1.2 The perspective of Kaput

Kaput's agenda is quite different. He is explicitly interested in curricular implications of historical investigations of calculus. Indeed, his chapter is divided into three parts and he describes the second, which makes up about 60% of the whole as a "curriculum- and pedagogy-sensitive historical overview of calculus" (p. 2). He is clear from the beginning about his goal for this study. On the first page he announces his intention to "...look closely at the origins of the major ideas of the calculus for clues regarding how calculus might be regarded as a web of ideas that should be approached gradually, from elementary school onward in a coherent school mathematics curriculum..." and to "...look closely at dynamic graphical means for representing important calculus ideas in ways that reflect their origins...". He appears to feel successful in establishing these two points because at the beginning of the third part of the chapter he presents them as "...recommendations drawn from the historical review and a survey of current conditons and possibilities..." (p. 42).

It is clear from this and from the rest of the paper that Kaput is making a cognitive application of "ontogeny recapitulates phylogeny" although he cautions against possible excesses in applying such a tempting principle (p. 5).

Another important way in which Kaput differs from Piaget & Garcia is in the role given to notational systems. Kaput considers them to be very important and devotes one (albeit the shortest) of his three parts to them. Piaget & Garcia hardly mention them at all. I believe that Kaput's views on notational systems and representations, which pervade his entire chapter, form a keystone of most of his thinking about epistemology. In Section 3 below, I will try to show how his chapter exemplifies the central role he gives to notation and also discuss some alternative points of view.

2 The Relation between History and Curriculum Design

Kaput's chapter includes a study of the history of calculus, what he calls a "longitudinally coherent calculus curriculum", and implications from this history which justify his curricular recommendations. I will discuss separately his history and the recommendations he draws from it.

2.1 Kaput's history of calculus

In his chapter, I believe that Kaput has made an important contribution to the study of the history of calculus. He begins with three root aspects of calculus. He then tries to find sources for them and trace their developments in various periods and movements of history.

2.1.1 The roots

Here are the roots he considers.

- Geometric issues related to computations of areas, volumes, and tangents.
- A mix of practical and theoretical interest involving the characterization and theoretical exploitation of continuous variation of physical quantities.
- Inherently theoretical concerns with the foundations of calculus.

2.1.2 A Piagetian analysis of the roots

Before describing briefly the extent to which Kaput sees these roots in different historical periods and in the work of various individuals and movements, let me consider for a moment the roots themselves and their possible connection with the perspective of Piaget & Garcia.

Kaput proposes such a connection when he suggests (p. 19) that "...the operant nature of the algebraic symbol system provided the means for the transition from the former stage (intra-object) to the latter (inter-object).." I think that Piaget would disagree, and I would like to offer an alternative.

The first root is clearly intra-object (as Kaput suggests) in that it is concerned with investigation of individual objects such as areas, volumes, tangents, etc. The key to the second root is variation of these objects, or certain qualities of them (such as size, position, etc.) Again, as Kaput points out, this is already at the inter-level in that it is concerned with relations among objects, or the same objects with different qualities (e.g., a body at one position compared with the same body at another position). Finally, he suggests that the transition from the first to the second is made by means of the algebraic symbol system and this is where I think that he is wrong, at least as an interpretation of Piaget, who states in several places his contention that the means of transition is reflective abstraction. See Dubinsky (1991a, 1991b) for specific refeences on this point.

More specifically, I would suggest that the main step in passing from intra- to inter- with reference to a particular quality of an object, such as its velocity, is the *encapsulation* of that quality which (as Kaput points out) is initially a process (e.g., change in position coordinated with change in time for the case of velocity). Only by thinking of something as an object is it possible to compare it with something else, or itself at a different time.

Similarly, I would suggest that the transition from inter-object to trans-object is done by means of imagining all possible processes (including their reversals and compositions) involving the objects of concern and coordinating them in a single totality or structure.

As I will try to argue below, the usefulness or even possibility of a symbol system is the result of, not the means of, the transition from intra- to inter-.

2.1.3 Finding the roots in history

Kaput looks for the roots and traces their development from the time of the ancient Greeks, through Oresme and the Scholastics in the pre-Renaissance period, the 17^{th} century, Newton and Leibniz, Euler and up to the present day.

Kaput explains (pages 8-9) that the Greeks did not make much progress towards the development of calculus, indeed could not even get started, because they were unable to quantify motion. In connection with this, they did not really have a conception of variable. To be sure, they used literals, but for them, a literal was nothing more than a place marker for an unknown constant, little different from a number.

Kaput gives great importance to Oresme and the Scholastics (pages 9-18). He suggests that their attempts at "...mathematizing genuinely experienced variation before algebra"(p. 41) is not only a root of calculus, but something we should encourage in students at a very early age. Several examples are given of attempts to represent various qualities, such as temperature, velocity, acceleration, time, and distance as geometrical objects such as lines and rectangles. Kaput points out several instances of children making similar attempts.

In keeping with his view of the role of notation, Kaput considers that it is the development, in the 17^{th} Century, of algebraic representations of variables via coordinate systems and via expressions that could be manipulated according to formal rules that made possible the calculus invented by Newton and Liebniz in the following century (p. 18).

An important next step, taken by Newton, is the reliance on motion imagery to conceptualize continuous phenomena. In the third part of his chapter, discussed below, Kaput tries to use video technology to implement this development.

Kaput emphasizes Leibniz' contribution in inventing a powerful symbolic notation for differentiation and integration. He also mentions the role of finite difference calculus from which comes the use of discrete formulations to model continuous phenomena.

Finally in considering the period beginning with Euler and continuing to the present day, Kaput describes "...the shifts in both the semantics and the nature of justification as the roles of geometry and algebra shifted in the period between the 16^{th} and 19^{th}

centuries..." (p. 41) and discusses the movement to put the principles of calculus on a firm, logical foundation. For this last period, Kaput presents a relatively conventional description of the movement in the last two centuries to work out the foundations of calculus.

2.1.4 On Weierstrass' $\epsilon - \delta$ definition

Before leaving the discussion of the historical development of calculus, I would like to offer an alternative to Kaput's rather standard interpretation of Weierstrass' definition of limit. According to Kaput, Weierstrass' $\epsilon - \delta$ definition "...was a static concept, replacing the motion metaphor used in Cauchy's definitions...no longer did values of a variable 'approach indefinitely a fixed value'. Weierstrass rendered limits atemporal concepts".

I would put it a little differently. I would like to suggest that, contrary to what most people believe, Weierstrass' definition is *not* static. The dynamics are still present in his notation, but they have been moved from "external" metaphors to "internal" mental processes. Or to put it more precisely from a constructivist point of view, a person's understanding of notation is a coordination of what is present in the notation (e.g., the marks on the paper) and what is present in the mind of the operson. Kaput refers to this as "...the act of 'building meaning' from the notations" (p. 3).

In the mathematical concept of limit, there is, intrinsically, dynamic processes. One variable is moving, in its values, along a path in its domain towards some particular value. Simultaneously, a second, dependent, variable is moving in some manner within its domain. The relationship between the variables is a coordination of two processes. Up until Weierstrass, the goal was to capture the processes and their coordination *in the markings of the notation*. This is really not possible, at least with symbols and writing and maybe not even with videos and computers, since even videos can only represent finite processes. With Weierstrass, there is, for the first time, a realization that these processes must be constructed in the mind which *can* handle temporal change and is the only tool we know that can be used to construct an infinite process. The notation is then relegated to the role of a support for the construction of the mental process.

Moreover, in this formulation, the notation (that is, the traditional $\epsilon - \delta$ notation), if viewed on its own, *does* appear to be static and devoid of meaning — empty formalism according to many. It is only when the notation is read together with appropriate mental processes *in the mind of the reader* that the concept takes on its true dynamic nature. Thus it is no surprise that, for someone who has not or is not about to construct a certain pair of mental processes and coordinate them in a particular way, this notation is a piece of empty formalism and remains totally incomprehensible. The important point is that meaning resides in a *combination* of what appears in the symbolism and what is constructed in the mind of the reader of that symbolism.

The main pedagogical implication here is that symbolism itself does not convey ideas. It is necessary for instruction to be aimed at helping students make appropriate mental constructions which they can combine with powerful representations in order to build their mathematical concepts.

2.2 Kaput's recommendations

As a consequence of his historical analysis, Kaput draws two major conclusions which he expresses in the form of the following two recommendations (p. 42).

1. Calculus needs to be studied across many years of school, from early grades onward, much as a subject like geometry should be studied. Hence its many <u>purposes</u> should be examined, not merely its refined methods. But most especially, its root problems should take precedence as the organizing force for curriculum design.
2. The power of new dynamic interactive technologies should be exploited in ways that reach beyond facilitating the use of traditional symbol systems (algebraic, numeric, and graphical), and especially, in ways that allow controllable linkages between measurable events that are experienced as real by students and more formal mathematical representations of those events.

2.2.1 A longitudinally coherent calculus curriculum

In his first recommendation, Kaput is really getting at an issue which is the essence of the idea of academia in our culture. From time to time, there occurs in human endeavour, a relatively coherent set of ideas, such as calculus. It is important to make those ideas part of our culture and hand them down to succeeding generations. This is one of the purposes of our schools. The only way we know how to do this is to create academic courses. Unfortunately, these courses, by their very nature, are static, unresponsive to local needs and tend to concentrate, somewhat unrealistically, on the end product, rather than the process(es) by which it came about.

Kaput calls for a reexamination of the tacit consensus that calculus is what he calls a capstone course. He would like to see the ideas of calculus spread out through the total education of all students. It is difficult to disagree with this goal but much harder to see how to bring it about. As Kaput readily admits, the present educational system is not very conducive to absorbing calculus in this way. It is a little disappointing that Kaput tells us very little (pp. 42-43) about what calculus might look like in a curriculum that followed this recommendation, and he tells us nothing about how one might actually bring about such a new curriculum.

One thing is, however, very clear about the curriculum Kaput is calling for. It should strongly reflect the root problems which, his analysis suggests, were the driving force in the historical development of calculus. It is possible to argue with this suggestion.

Let us grant that the root problems are those given by Kaput and that they drove the historical development of calculus. Why should they drive the development of the ideas of calculus for students today? Perhaps we don't need to aim cannons to fly over the walls of medieval castles. Perhaps we don't (or shouldn't) need to map out the trajectories of intercontinental missiles. Perhaps we have a greater need to analayze DNA molecules or understand the behavior of the market. In the latter case, we might wish to reverse the traditional use of calculus in which the discrete is an approximation of the continuous and study fluctuation of prices in which the continuous is a model for the discrete.

In other words, the root problems come from the needs of society. If those needs change (or should change) what are the implications for calculus curricula? Kaput does not deal with this question. He assumes tacitly that the historical root problems form the only alternative.

2.2.2 A motion world learning environment

Most of the last part of Kaput's chapter (pp.44-60) is related to his second recommendation and consists of a description of a video system *MathCars* that uses dynamic, interactive video technology to simulate driving a car. The basic idea of the system is "...to map the phenomenologically rich experience of motion in a vehicle (sights and sounds) onto coordinate graphical and other mathematical notations." The user views a video screen which presents the motion from various perspectives. He or she controls an accelerator and can coordinate representations of time, distance travelled, or velocity. These representations, in the form of graphs, can be studied as objects in their own right.

The descriptions and pictures presented are very interesting (and are even more exciting when viewed on video as was the case at the conference where this chapter was presented). It will be even more interesting to see how this system can be used in the kind of curriculum Kaput is proposing and to hear about results that are obtained. Hopefully, this is on Kaput's agenda for the not too distant future.

2.3 Justification

Perhaps the most serious weakness in Kaput's chapter lies in the connection between his historical analysis, which, as I have indicated, is quite profound, and his recommendations, which certainly *must* be on the table as the revamping of mathematics education proceeds. I do not think that Kaput has really made a strong case for his contention that the recommendations follow from, or are even justified by the historical analysis. His case is restricted to a relatively small number of uncoordinated examples in which histroical difficulties are mirrored by students. I feel, however, that Kaput does not argue persuasively either that his conclusions are justified by his historical analysis, or that these same conclusions can not be obtained from other analyses.

We have already indicated above one argument that accepts Kaput's root problems but suggests an alternative conclusion. Here is another. It is possible to conclude from the historical analysis, especially the many centuries it took for certain ideas to emerge, that the concepts of calculus are very difficult and therefore should not be introduced earlier, but later! Instead, let the period from kindergarten up to, say, the second year of college be spent in *preparing* students for calculus. In my opinion, most of that time should be spent with the various aspects of functions, but that is another discussion.

Even if one accepts Kaput's conclusions, it is not clear that the historical analysis is needed to justify them. Indeed, Kaput himself presents (pp.42-44) several arguments not based on his historical analysis that are aimed at justifying his proposed curriculum. Moreover, it is somewhat jolting, in this age of technological explosions in education, to imagine that Kaput's second recommendation needs any justification. It is as if someone were standing in the middle of a hurricane trying to argue that the wind should be blowing.

3 Notation and Representation

Kaput begins his discussion of notation with the following statement (p. 3) (parenthetical statements added, making use of Kaput's subsequent qualifiers for the specific case of mathematics).

> "I take the point of view that we organize the flow of (mathematical) experience jointly using two structures, one mental — the structures of mind (mathematical knowledge) — and the other material — the material artifacts (mathematical notations related to the knowledge), including spoken and written language, produced and used in accordance with our cultural inheritance in one or more physical media."

No one can argue with this statement, in particular the joint (and, presumably, balanced) role of the two structures of conceptual understanding and formal symbolism. Nevertheless, a "chicken and egg" question is inevitably raised. Does one of these two structures dominate the other? Is it the case that one is made possible by the other which cannot otherwise progress?

One quote of Cajori that is given by Kaput (p. 1) does suggest such an asymmetry.

> "Without a well-developed notation the differential and integral calculus could not perform its great function in modern mathematics."

Others have tilted matters in the opposite direction as did Tolstoy when he suggested that there is often difficulty in learning a new word, not because of its sound, but because of the concept to which the word refers. He concluded as follows.

> "There is a word available nearly always when the concept has matured"[2]

There certainly is an issue here and I will first argue that, although in his other works Kaput may discuss the construction of concepts, in this paper at least, he really does appear to emphasize the idea that notation must precede, if not dominate conceptual understanding. I will then try to present an alternative point of view, which seems more reasonable to me.

[2] L. Tolstoy *Pedagogicheskie statli* [Pedagogical Writings], Moscow: Kushnerev, 1903.

3.1 The role of notation in Kaput's chapter

First I will try to show that Kaput's chapter presents a point of view about the relation of conception over notation that is quite unbalanced. Then I will discuss his idea of Action Notation which is a very important notion. Although I will later present an analysis of this idea that is different from Kaput's, I certainly agree with his description of its role in the development of mathematics.

3.1.1 Predominance of notation over conception

I base my contention that Kaput's chapter presents an unbalanced view on the fact that I find, in the chapter, a large number of specific places in which he expresses this dominance. I will list a few examples and leave it to the reader of this volume to see if I am right in suggesting that there are very few examples in the other direction.

- Action notation "had profound impact on the nature of mathematics" (p. 5).
- "We mention the terminology to give a sense of the difficulty experienced by those who are attempting to develop a coherent mathematical theory of variation before a systematic language for the expression of a theory is available." (p. 9)
- "The Scholastic philosophers were striving to express their ideas in words and geometrical diagrams, and were not so successful as we who realize, and can make use of, the economy of thought which mathematical notation affords." (p. 16)
- In reference to Descartes, Kaput writes "...his symbolism for exponents...led to a new conception of number and variable that opened a whole new world of possibilities." (p. 20)
- Although he opens his section on Leibniz with the balanced statement about "...the development of notation in concert with the development of concepts", he titles this section "Leibniz: The Power of Notations" (p. 27). Nowhere does Kaput discuss, much less use as a title, *the power of concepts* to engender notations!
- He refers to a "...pattern of syntactically driven extensions of symbol systems leading to conceptual extensions, in particular, to the construction of conceptual entities" (p. 32).
- In considering functions, Kaput suggests the following causal relation, "Euler's general definition was not applied even by him, and then for a century hence, because there was no way to represent — and hence meaningfully study, compute or apply — such general objects." (p. 37).

I am not suggesting that I think Kaput is wrong in all of these statements. I think that several of the above points have some truth to them. I present them because, as a whole, they (and the absence of very many balancing points) suggest to me that Kaput really does believe that you can develop a notation and then this can lead to understanding a concept or that understanding a concept must wait for the development of appropriate notation. I will try to show below how at least some of the above points can be interpreted reasonably by assuming that it is the other way around.

3.1.2 Action notation

Kaput makes a useful contribution in analyzing notational systems when he divides them into two kinds. A *display system* serves "primarily to display information for the user to read/respond to", whereas an *action system* provides "systematic means for the user to act on it physically" (p. 18). Examples of the latter, which are clearly the more important, abound: algebraic expressions, use of 0 in writing integers, $\frac{dy}{dx}$, etc.

No one could disagree with the very strong case that Kaput makes (pp. 18-24) for the critical importance of action notation as an indispensable tool for making progress in mathematics. It is so valuable that it (rightly) maintains it pre-eminent position among mathematical devices in spite of the fact that there are two major drawbacks involved with using action notation.

One of these is pointed out by Kaput (p. 41) in terms of Skemp's notion of *instrumental understanding* as opposed to *relational understanding.* The very power of many examples of action notation creates the possibility (unfortunately too often realized in our schools and colleges) that students will learn to perform mechanical operations without much understanding of what is behind them. This problem is so serious that we actually have a society which is convinced that the manipulation of symbols is the essence of mathematics and that mathematical brilliance consists of multiplying or factoring very large integers. (The recent film, "Little Man Tate", whose director and star is a Princeton graduate, is a case in point. It depicts a young boy who is purported to be a mathematical prodigy. With one exception, everything that this child does consists of mentally performing arithmetic caclulations with large numbers very quickly.)

The second drawback is that as the concepts get more complicated, and as the portion of the population that must be skilled in mathematics grows, it becomes more and more difficult to get students to be successful, even with the manipulation of symbols — especially when it is not completely rote. When unusual letters are used in familiar situations, things fall apart. I have seen graduate students become disconcerted when asked to explain what is meant by $(x+y)(F) = x(F) + y(F)$. And how many of us who teach calculus have seen students insist that the derivative f' of the function f given by $f(x) = a^x$ is given by $f'(x) = xa^{x-1}$? Thus we have two problems in student performance with notation. One is that it can be hard for them to develop skills and second, they can become wedded to very specific choices in notation, unable to function with even the slightest variations.

3.2 An alternative to Kaput's view on notation

First I will point out some questions that arise and are not, in my opinion, dealt with sufficiently in Kaput's use of notation. Then I will present a different point of view, use it to reformulate some of Kaput's specific comments, and indicate very briefly how it might be applied pedagogically.

3.2.1 Some difficulties with Kaput's idea of notation predominance

Kaput's historical analysis does not tell us anything about where notation come from. We learn that there are instances in which long waits were necessary (centuries even) before appropriate notational schemes were devised (p. 9). But there is nothing about what causes the delay, or what sorts of developments had to take place in order for effective notation to arise.

Kaput refers to the "real achievement...of the masters who built the concepts initially and wrote them in language that embodied the organizing syntax that we consumers can use with confidence" (p. 40). Is he telling us that progress must wait for geniuses? And what about our students, or at least those whose reaction to various notational schemes is something other than "use with confidence"?

This is another, related difficulty. Since the historical analysis doesn't tell us about where notational schemes come from or how they arise, it is not very helpful in pointing to pedagogical solutions to students' notational *and* conceptual difficulties.

Now I am fully aware that Kaput does, in this paper and in many other works, propose approaches, using computers and other forms of technology to help students develop representations of mathematical concepts. What I am suggesting is that although this historical analysis convinces us that we should not be surprised at certain difficulties of our students, it tells us very little, in principle, about how to deal with those difficulties.

3.2.2 Predominance of conception over notation

Therefore, I would propose an alternative point of view, in which construction of concepts comes before development of notation. Kaput points out (p. 31) that several people (including this author) have worked with the idea of mental construction of objects. For me, the construction of mathematical concepts includes a very important step that takes place after the construction of a mental process. It is the *encapsulation* of that concept into an object. What is suggested by the research of myself and others is that first one encapsulates a process into an object. This is very difficult. It is what takes time and for which one must devise special instructional treatments, in which actions are applied to processes. Once it has been accomplished, however, a notational scheme can be developed *and connected to the concept* by the relatively simple act of *associating* a syntactically governed set of symbols with a mental object that an individual has already constructed.

That is, to paraphrase Tolstoy's comment, I would say that a notational scheme can be learned and used effectively by an individual, with understanding, once the concepts to which this scheme refers have been constructed, by that individual, as mental objects or entities.

As I indicated earlier, it is this act of encapsulation that is the key mechanism in the transition from the intra-object level to the inter-object level in the sense of Piaget & Garcia.

3.2.3 Some alternative explanations

Let me consider a couple of examples.

Kaput writes about the struggles of the Scholastics to find reasonable ways to represent acceleration, and the fact that they did not succeed. From the theoretical perspective just introduced, one could argue that before it is possible to construct the concept of acceleration, one must first develop an understanding of velocity as a process of comparing distances at different times. Then this process is encapsulated to a conceptual entity so that velocity becomes a mental object. Only then is it possible for velocity to have different values at different times, so that one can go on to the idea of acceleration.

Or consider the long wait before Euler's general definition of function became useable (p. 37). I would suggest that the wait was not, as Kaput suggests, for ways of *representing* functions as objects, but for people to develop the ability to encapsulate function processes as objects. Indeed, if Kaput were right, why do we have so much trouble getting students to work with functions as objects (the derivative of a function is a function, the solution of a differential equation is a function, the composition of two functions is a function, etc.) We have an excellent notation to show them, but it doesn't help very much. They still must struggle to make encapsulations.

3.2.4 Pedagogical applications

At the very least, we can use this alternate formulation to propose, implement, and evaluate instructional methods for helping students develop various concepts which require the construction of mental objects. We can do this using computer systems in which students can write programs that accept functions as inputs and produce functions as objects, that provide the opportunity to perform actions on certain concepts which are to be the referents of notational symbols. See Ayers et al (1988) and Breidenbach et al (in press) for examples which indicate some success in this endeavor.

These initial indications of success are a far cry from establishing the validity of the theoretical perspective I am proposing, or its preferability over Kaput's point of view. It does suggest, however, that it might be a viable alternative.

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