The Development of Students' Graphical Understanding of the Derivative[1]

Mark Asiala
Purdue University
W. Lafayette, Indiana

Jim Cottrill
Purdue University
W. Lafayette, Indiana

Ed Dubinsky
Purdue University
W. Lafayette, Indiana

Keith E. Schwingendorf
Purdue University North Central
Westville, Indiana

Abstract

This paper is part of a series of studies by the "Research in Undergraduate Mathematics Education Community" (*RUMEC*), concerning the nature and development of college students' mathematical knowledge. The present study explores calculus students' graphical understanding of a function and its derivative. An initial theoretical analysis of the cognitive constructions necessary for this understanding is given. An instructional treatment designed to help foster the formation of these mental constructions is described, and results of interviews, conducted after the implementation of the instructional treatment, are discussed. The understanding demonstrated by these students is analyzed according to the Action-Process-Object-Schema (APOS) theoretical framework. Based on the data collected as part of this study, a revised epistemological analysis for the graphical understanding of the derivative is proposed. Moreover, a comparative analysis is made of performance of students using the instructional treatment we designed with students taking a traditional calculus course. Although this analysis is flawed in many ways, it does suggest that the students whose course was based on the theoretical analysis of learning that we give here may have had more success in developing a graphical understanding of a function and its derivative, than students from traditional courses.

1 Introduction

This paper reports on a study of the nature of calculus students' graphical understanding of the derivative. The study was carried out according to a research methodology that is being developed by the members of the Research in Undergraduate Mathematics Education Community (*RUMEC*), for the purpose of studying how collegiate mathematics can be learned. A brief summary of the *RUMEC* framework for research and curriculum development is given in Section 1.1 below.

The goals of the current study are as follows: to determine to what extent the Action-Process-Object-Schema (APOS) theoretical perspective (see Asiala et al., in press) is useful for under-

[1]Work on this project was partially supported by National Science Foundation Grant No. USE 90-53432 and a grant from the Exxon Education Foundation. Any conclusions or recommendations stated here are those of the authors and do not necessarily reflect official positions of the NSF or the Exxon Education Foundation.

standing the mental constructions made by students learning about the graphical interpretation of the derivative; to increase our understanding of how learning about the slope of a graph of a function might take place; to make a rough comparison of the performance of students in this study who took the Calculus, Concepts, Computers, and Cooperative Learning, or C^4L, reformed calculus course (see Schwingendorf, Mathews, & Dubinsky, 1996) with students who took a traditionally taught (lecture/recitation) calculus course; and to develop a base of information which sheds light on the epistemology and pedagogy associated with the concept of derivative.

Our main findings are of two kinds. First, the theoretical perspective does appear to be useful in describing the development of graphical understanding of a function and its derivative. Our analysis suggests that it might be helpful for students to study these concepts in courses using pedagogy based on the APOS perspective. Moreover, our second finding, albeit very preliminary, is that when students take courses designed in this way, they may come to better understandings than if they took traditional calculus courses.

After summarizing our framework for research and curriculum development, reporting on the literature on these topics, and describing the methodology of this study, we present our results, revise our initial genetic decomposition, and make suggestions for pedagogical strategies and future studies.

1.1 Summary of framework for research and curriculum development

Our framework for research and curriculum development has three components: theoretical analysis, instructional treatments, and observations and assessments. We will describe our approach only briefly; the reader is referred to Asiala et al. (in press) for a complete discussion of the framework.

Theoretical analysis. The first step in this approach is to make an initial theoretical analysis using our theoretical perspective on learning theory, the epistemology of the concept being studied based upon past research, literature, and the mathematical knowledge of the researchers. The purpose of the theoretical analysis is to propose a *genetic decomposition* or model of cognition: that is, a description of specific mental constructions that a learner might make in order to develop her or his understanding of the concept. These mental constructions are called actions, processes, objects, and schemas, so that the theoretical framework we use is sometimes referred to as the APOS Theory.

According to APOS theory, an *action* is a transformation of mathematical objects that is performed by an individual according to some explicit algorithm and hence is seen by the subject as externally driven. When the individual reflects on the action and constructs an internal operation that performs the same transformation then we say that the action has been *interiorized* to a *process*. When it becomes necessary to perform actions on a process, the subject must *encapsulate* it to become a total entity, or an *object*. In many mathematical operations, it is necessary to de-encapsulate an object and work with the process from which it came. A *schema* is a coherent collection of processes, objects and previously constructed schemas, that is invoked to deal with a mathematical problem situation. As with encapsulated processes, an object is created when a schema is *thematized* to become another kind of object which can also be *de-thematized* to obtain the original contents of the schema.

Instructional treatments. Genetic decompositions of mathematical concepts are the basis for the design of the instructional treatments. The pedagogical approach used to implement the instructional treatments for helping students make the proposed mental constructions is called the ACE teaching cycle (**A**ctivities, **C**lass tasks and **E**xercises). The main strategies of this method include having students construct mathematical ideas on the computer using a mathematical programming language, investigate mathematical concepts using a symbolic computer system, and work in cooperative learning groups to engage in problem solving activities and in discussions of the results of the computer and problem solving activities.

Observations and assessments. Implementing the instruction provides an opportunity for gathering data. There are two ways in which these data are related to the genetic decomposition. First, the genetic decomposition directs the analysis of data by asking the question: did the proposed mental constructions appear to be made by the students? Second, the results of this analysis may lead to revisions in the genetic decomposition, which may lead to changes in the instructional treatment. The cycle is then repeated as many times as is necessary for the researcher to come to a deeper understanding of how students can construct their understanding of the concept. Initially, the genetic decomposition is based primarily on the researchers' own understandings of the concept, and on their experiences as learners and teachers. As the cycle is repeated, however, the genetic decomposition comes more and more to reflect the analysis of data. In each repetition of the research cycle, additional data are gathered to report on the performance of students on mathematical tasks related to the concept in question. The analysis of this additional data tries to determine directly what mathematics may have been learned, rather than what mental constructions might, or might not have been made.

In other words, we collect different kinds of data. Some of it is analyzed to see what mental constructions the students might have made. If their constructions are what is required by the genetic decomposition, then the theory predicts that mathematics will have been learned. A second kind of data is analyzed to see if this prediction is supported. In this way, both our theoretical analysis and our instructional treatments are evaluated.

Thus the outcome of work under this framework is, by its nature, two-fold. One result of the research is the deepening of our understanding of the epistemology of the concept. The second result is the creation of pedagogical strategies which are better aligned with the way we believe that students come to understand the concept; these improved strategies should thus lead to increased learning by the students. The overall organization of this paper reflects this dual nature with the presentation of both epistemological findings and pedagogical findings along with the relations between them.

1.2 Some comments on the literature

Although it is generally assumed by calculus instructors that students coming to study this subject will bring with them a sufficiently strong concept of function, the present study supports the observations of several authors that this is not the case for many students. Hence, we must indicate something of what the literature says about understanding functions, beginning with reports about this concept in a general context.

Many researchers have found that students have a limited view, that is, a weak concept image (Tall & Vinner, 1981) of function. Moreover, it is reported that students exhibit a predominant reliance on the use of and the need for algebraic formulas when dealing with the function concept (for example, see Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dreyfus & Eisenberg, 1983; Eisenberg, 1992; Schwingendorf, Hawks, & Beineke, 1992; Tall & Vinner, 1981; Vinner, 1989; and Vinner & Dreyfus, 1989).

Tall and Vinner (1981) suggest that student difficulties in dealing with a function given in graphical form may be a result of traditional instructional methods. They suggest that although the general notion of function, which includes functions defined by graphs, may be utilized for a short time, the predominant use of functions given by algebraic formulas may contribute to the development of a restricted notion, or concept image, of function. This has been verified by a number of researchers, including Leinhardt, Zaslavsky, and Stein (1990) and Lauten, Graham, and Ferrini-Mundy (1994). Leinhardt et al. attribute part of the difficulty students have in connecting functions and their graphical representations to traditional instructional methods that tend to emphasize the construction of graphs from algebraic formulas or tables, but do little in the way of the reverse — from graphs to algebraic formulas or tables.

We turn now to what the literature says in the context, closer to our situation, of understanding functions that arise in calculus.

The limited conceptual view of the function concept held by students passing through traditional calculus sequences has been documented in various studies (Orton, 1983; Selden, Selden, & Mason, 1994). In particular, Selden et al. (1994) report how even students who performed very well on routine calculus problems found great difficulty and had little or no success in dealing with problems that were non-routine.

Breidenbach et al. (1992), Sfard (1987, 1992), Thompson (1994) and others make a case for the the development of a strong process conception of function prior to an object conception in the development of the students' understanding of the function concept. It is believed that a lack of, or weakly developed, process conception of function contributes to student difficulties in dealing with the function concept. These authors and others also discuss the importance of the notion of a function as an object, or total entity, which is the result of encapsulation (see Breidenbach et al., 1994), or as Sfard puts it, the reification of processes into (mental) objects. This can be very difficult. Sfard makes the point that neither the algebraic nor "the solidifying" graphic representation of function are very effective as a means of converting function processes into objects. On the other hand, positive results on the conversion of the process conception of function into an object conception through the writing of computer functions can be found in the paper by Breidenbach et al.

Finally, we describe briefly some of the literature on understanding the derivative and relating it to a graphical situation, and investigations about slope.

Many authors discuss specific problems students have with the graphical interpretation of the derivative. For example, Ferrini-Mundy and Graham (1994) describe a number of difficulties students have with basic calculus concepts. They point to the work of Amit and Vinner (1990) which reported that some students equate the derivative of a function with the equation for the line tangent to the graph of the function at a given point. This phenomena was also found to occur with some students in the study considered in this paper.

Ferrini-Mundy and Graham (1994) discuss in detail one student's desire to find an equation for a function represented graphically before trying to sketch the graph of the derivative.

Orton (1983) noted that students' routine performance on differentiation items was adequate, but they had little intuitive or conceptual understanding of the derivative concept. Orton also reported that student difficulties with graphical interpretations of the derivative can occur in the case of straight lines, not only with more complicated curves. He also found that many students (about 20% in his study) confused the derivative at a point with the ordinate, or the y-value (second coordinate) of the point of tangency.

Leinhardt et al. (1990) review papers reporting that students had difficulty in computing slopes from graphs. As noted above, they call for a greater emphasis on the reverse — from graphs to algebraic formulas or tables in an effort to help students deal with difficulties they encounter with graphical problems.

Eisenberg (1992) reports on the students' "avoidance to visualize" and hence concurs with the conclusion of Vinner (1989) mentioned above. In addition, Eisenberg presents a test of 10 problems that he claims most calculus course graduates will fail, partially due, in his view, to the students' lack of a "sense of functions". We note that the first question on Eisenberg's exam is almost identical to our Question 6 in the clinical interview which is the primary basis for the study reported on in this paper. We also note that Eisenberg did no in-depth clinical interviews in his study.

Tufte (1989) reports on a study of 200 first year college calculus students' performance on a "Calculus Concept Test" which included a problem almost identical to our Question 6. In the same paper, Tufte also reports on a follow-up study of 24 first semester calculus students who completed a calculus course in which they met one additional class period each week to write programs on basic calculus concepts (e.g., to approximate limits, derivatives, and definite integrals via Riemann sums). The results of the follow-up study suggest that a programming experience which involves the writing of computer code on basic concepts can have a profound effect on students' conceptual understanding as compared to that of students who complete a traditional calculus class. We note that the results of Tufte parallel the positive results we report on in this paper for C^4L calculus students as compared to traditional students. Our analysis is an attempt to further understand the nature of this salutary effect, in terms of mental constructions, of writing programs. We also note that Tufte analyzed students' written responses to the test questions and he did not do any in-depth clinical interviews as we did in the study reported on in this paper.

Finally, we mention the major study of student understanding of the concept of slope by Schoenfeld, Smith, and Arcavi (1990). In this paper, the authors make a long term, very fine-grained, study of one student's construction of the concept of slope. They find that what appears to most people working at any level in mathematics to be a very simple concept can present serious difficulties — even to students that are reasonably strong.[2]

2 Description of this study

2.1 Students

The student population for our study was 41 engineering, science and mathematics students who had previously taken (at least) two semesters of single variable calculus at a large midwestern university during the time period from Fall of 1988 through the Spring of 1990. Of these 41 students, 17 had completed at least one semester of the C^4L courses. The remaining 24 students completed a two semester calculus sequence taught as a traditional lecture/recitation course with lectures to about 400 students (one student completed a two semester advanced placement calculus sequence which also covered multivariable calculus taught as a traditional lecture/recitation,

[2] Although the theoretical perspective, research questions and results are very different from ours, we consider our work related to the work of Schoenfeld et al. in that both point out, and attempt to understand, the difficulties students can have with the concept of slope.

but with only about 30 students).

The students were selected for this study informally and no attempt was made to randomize their characteristics. We did attempt to look at how the C^4L students in our population compared with the students from traditional classes in terms of their overall records and past academic performances. The results of this comparison will be reported later in Section 3, **Results**.

The students were paid at a nominal rate for their participation. Interviews were conducted and audio-taped. Each interview lasted for about an hour, and the tapes were transcribed by paid student aides and then proof-read.

2.2 Interview questions

Each student in this study completed an "Interview on Derivatives" given by some of the researchers in this study and other members of *RUMEC*. Each clinical interview consisted of eleven questions on the concept of derivative.

For the interview questions on which this study is based, each student was given a sheet of paper containing the figure in Question 6 below and then asked the following question:

Interview Question 6. Suppose that the line L is tangent to the graph of the function f at the point $(5,4)$ as indicated in the figure below[3]. Find $f(5)$, $f'(5)$.

quest6.eps scaled 1000

Figure 1: Question 6

Each student was asked "Explain how you arrived at your answers."

The student was given another sheet of paper and asked the following question:

[3]The reader will notice that there is an inaccuracy in this graph that could lead to computing the slope of the line as 2/5 or 4/9. From all indications when students were asked about this, it did not seem to lead to any confusion.

Interview Question 7. Sketch a graph of the function h which satisfies the following conditions:

$$h \text{ is continuous}$$
$$h(0) = 2, \; h'(-2) = h'(3) = 0, \text{ and } \lim_{x \to 0} h'(x) = \infty,$$
$$h'(x) > 0 \text{ when } -4 < x < -2, \text{ and when } -2 < x < 3,$$
$$h'(x) < 0 \text{ when } x < -4, \text{ and when } x > 3,$$
$$h''(x) < 0 \text{ when } x < -4, \text{ when } -4 < x < -2, \text{ and when } 0 < x < 5,$$
$$h''(x) > 0 \text{ when } -2 < x < 0, \text{ and when } x > 5,$$
$$\lim_{x \to -\infty} h(x) = \infty \text{ and } \lim_{x \to \infty} h(x) = -2$$

While the student was working on the graph, and after the graph was finished, he or she was asked to explain everything that was done. Also, each student was asked "What other kinds of curves are possible?"

Question 7 was used to see what information in the student responses could further explain the student's graphical understandings that we observed in Interview Question 6.

This paper attempts to make a careful study of Interview Question 6 which addresses the graphical interpretation of the derivative, and a partial study of Interview Question 7 as it pertains to the geometric (or graphical) aspect of the derivative.[4]

2.3 Methodology

We now describe our methodology. Each of the four researchers involved in this study did a preliminary analysis of a subset (10-11 interviews) of the 41 transcripts with the following specific questions in mind:

- How does a student understand the derivative of a function at a point graphically?
- Is this connected to the student's graphical understanding of the value of a function at a point?
- How does this correlate with whether the student took the C^4L or a traditional calculus course?

Then, each researcher carefully analyzed each of the 41 transcripts of Interview Questions 6 and 7 with eight specific questions in mind, assigning a numerical code that rated each student on each question. This was done independently and differences were negotiated so that all researchers

[4] Analysis of the other interview questions will be included in other *RUMEC* studies on calculus concepts.

agreed on the final numbers that were assigned. It should be noted that these numbers came from the transcripts of the interviews and represent interpretations made by the researchers, expressed numerically for convenience. The eight questions were divided into two groups: one group of four questions on graphical understanding of a function and another group of four questions having to do with graphical understanding of its derivative. Here are the eight questions:

Graphical understanding of a function

F1 Did the student appear to understand $y = f(x)$ notation as it was presented on the graph?

F2 Was the student able to deal with the function based only on graphical information and without making use of an expression that defines the function?

F3 Did the student appear to understand functional notation in general?

F4 Was the student successful in drawing a graph of the function from specific information given about values of the function and the derivative (Question 7)?

Graphical understanding of the derivative of a function

D1 Did the student appear to understand that the value of $f'(x)$ is the slope of the tangent to the graph of the function at the point $(x, f(x))$?

D2 Was the student able to deal with the derivative of the function based only on graphical information and without making use of an expression (for example differentiating it) that defines the function?

D3 In working on Question 7, did the student appear to understand that as the value of the derivative approaches infinity, the slope of the graph of the function also grows without bound and hence there is a vertical tangent line to the graph of the function?

D4 In working on Question 7, did the student appear to understand how to use the derivative to determine intervals of monotonicity for the function?

In going through the transcripts, we developed an elaborate coding system for each of these questions. The codes varied from 4 points to 7 points, depending on the variety of responses found in the interviews. This system was used to identify the mental constructions and to evaluate the performances of students in Section 3. For our purposes in this report, we simplified the coding to a three point system, which we give in Section 3.3, by merging categories in some questions and/or re-interpreting others.

2.4 Initial theoretical analysis — an initial genetic decomposition

In this section, we give our initial genetic decomposition of the derivative concept, the result of our initial theoretical analysis, which was the basis for the instructional treatment given to students in the C^4L reform calculus courses. We assume that students come to a calculus course with the following prerequisite knowledge.

Prerequisite knowledge.

i. A working knowledge of and an understanding of the graphical representation of points on a curve in a coordinate plane together with the ability to connect points with the curve and interpret points from the curve.

ii. A working knowledge of and an understanding of the concept of slope of a line.

iii. A working knowledge of and strong understanding of the concept of function (hence a well developed concept image of function).

Graphical and analytic paths to the derivative.

We suggest that there are (at least) two related paths that can be traversed in constructing a schema for the concept of derivative: a graphical path and an analytic path. These are indicated in the next three steps of the following list. In the remaining steps, these two paths, which may not be totally disjoint, are coordinated.

1a. Graphical: The action of connecting two points on a curve to form a chord which is a portion of the secant line through the two points together with the action of computing the slope of the secant line through the two points.

1b. Analytical: The action of computing the average rate of change by computing the difference quotient at a point.

2a. Graphical: Interiorization of the actions in point 1a to a single process as the two points on the graph get “closer and closer” together.

2b. Analytical: Interiorization of the actions in point 1b to a single process as the difference in the time intervals get “smaller and smaller”, i.e., as the length of the time intervals get “closer and closer” to zero.

3a. Graphical: Encapsulation of the process in point 2a to produce the tangent line as the limiting position of the secant lines and also produce the slope of the tangent line at a point on the graph of a function.

3b. Analytical: Encapsulation of the process in point 2b to produce the instantaneous rate of change of one variable with respect to another.

4. Encapsulation of the processes in points 2a and 2b, in general, to produce the definition of the derivative of a function at a point as a limit of a difference quotient at the point.

5. Coordination of the processes in points 2a and 2b in various situations to relate the definition of the derivative to several other interpretations.

6. Interiorization of the action of producing the derivative at a point into the process of a function f' which takes as input a point x and produces the output value $f'(x)$ for any x in the domain of f'.

7. Encapsulation of the process in point 6 to produce the function f' as an object.

8. Reconstruction of the schema for the graphical interpretation of a function using the relation between properties of functions and derivatives.

2.5 Instructional treatment of C^4L students

The instructional treatment of the derivative took place during the first eight to nine weeks of the first C^4L calculus course. The design of this C^4L course took into account the initial genetic decomposition of the derivative discussed in the previous subsection. Students performed several computer activities prior to a one week unit focusing on the concept of derivative about the tenth week of the course. The pedagogical strategy used in the C^4L calculus course is the ACE Teaching Cycle. It consists of a combination of computer **A**ctivities (designed to help students make the mental constructions in the genetic decomposition), **C**lassroom tasks without computers followed by discussion (aimed at getting students to reflect on what they had done on the computer and could do with pencil and paper), and **E**xercises (to help students reinforce the knowledge they had constructed). All of the student work was done in teams of three, four or five students. In general, these teams, or small groups, were permanent throughout each semester. The derivative concept was, of course, reconsidered throughout the students' subsequent study of calculus. We note that the instructional treatment on derivatives for the 24 students from the traditional calculus classes took place during the first three to four weeks of the first semester calculus course.

There were five kinds of computer activities involving: (1) the usual topics of approximations of slopes and rates; (2) graphical investigations of the derivative concept; (3) the construction of a computer function which represents an approximate derivative function D together with variations of D to interpret the notions of one- and two-sided derivatives, and which "gets at" or represents the formal definition of derivative; (4) rules for computing derivatives; and (5) various applications related to slopes of tangents, graphs of functions, graphical interpretations of derivatives, and instantaneous rates of change.

We should note that one important aspect of the C^4L course is a major treatment of the concept of function beginning very early (in the second chapter, which is entirely devoted to functions) and continuing throughout the entire text. Previous research (e.g., Breidenbach et al., 1992; Dubinsky & Harel, 1992) suggests that having students write computer functions which implement, as processes, functions found in various problem situations can have a major impact on students developing a process conception of function. Furthermore, having students construct and use programs which have functions as inputs and outputs helps students toward their development of the ability to see functions as (mental) objects. Both in Chapter 2 of the C^4L text, and in other chapters of the text, students are asked to write numerous computer functions to represent, or implement, mathematical functions, and they also construct and use programs which have functions as inputs and outputs.

Following are more detailed descriptions of what the students did in these five kinds of computer activities.

1. Computer investigations of approximation.

Students wrote computer code to compute, for example, the average rate of change of a falling body over a small interval of time. They were able to use their code to investigate what happens as the length of the time interval takes on increasingly small values and to predict "ultimate values" for the average rate of change which turns out to be, of course, the derivative of a function at a point.

Such activities relate to points 1b through 3b of our initial genetic decomposition.

2. Graphical investigations of the derivative concept.

Values of the difference quotient for small differences were an important class of examples in all of the activities leading up to the concept of derivative in this course. For example, the students were asked to consider the following freezing point function F of a mixture of two chemicals A and B as a function of the mole fraction m of chemical A which is given by the expression,

$$F(m) = \begin{cases} 5m^4 - 23.6m^3 - 146m^2 + 129m - 17 & \text{if } 0 \le m \le 0.76 \\ 45.056(m - 0.76)^{\frac{1}{3}} - 11.98132480 & \text{if } 0.76 < m \le 1 \end{cases}$$

The students were asked to estimate the slope of the tangent to the curve of this function at $m = 0.76$ in two ways: first by using the computer to produce a graph and then measuring the slopes of secants; and second by using a computer algebra system to obtain a formula for the difference quotient using small differences. Students were then instructed to use the limit feature of a symbolic computer system to calculate the limit of these slopes of secant lines through the point $(0.76, F(0.76))$ to find the slope of the tangent line at this point.

The students also worked in the lab to write computer programs that computed difference quotients, a model of the limit process, and the equation of the tangent line to the graph of a function at a given point. These tools, which the students had constructed, were used to construct computer graphs of the tangent at various points on the graph of functions such as the freezing point function, and to compare the picture at smooth points and "corner" points.

These activities relate to points 1a through 3b of our initial genetic decomposition.

3. Computer constructions of the derivative concept.

As an alternative to a difference quotient with a model of the limit, students were asked to write and use a computer function which represents an approximate derivative D. After some struggle, most groups were able to write a computer function such as the following[5],

```
D := func(f);
        return func(x);
                  return (f(x + 0.0001) - f(x))/0.0001;
               end;
     end;
```

In the lab and in class, students discussed what happened when this function was applied to computer representations of functions such as the freezing point function `F` at various points. They also discussed the meaning of such computer expressions as `D`, `D(G)`, `D(G)(0.5)`, `D(G)(0.76)` for a given computer function `G`.

Finally, students are asked to write another version of the computer function `D` which is an "encapsulated" **ISETL** model of the formal definition of the derivative.

```
D := func(f);
       return func(x);
                if h /= 0 then dq := (f(x+h)-f(x))/h;
                return lim(dq,0);
                end;
              end;
     end;
```

Note that the computer function `D` uses a computer function `lim` that students have previously used in learning about limits and which is an **ISETL** model of the limit process.

In another related computer activity, students were also asked to implement the idea of two-sided derivatives by modifying their computer function `D` appropriately.

This work relates to point 4 of our initial genetic decomposition.

[5] The code presented here as examples is written in the mathematical programming language, **ISETL**. Because its syntax is so close to standard mathematical notation, it may not be necessary to provide explanations of it here. For further information and for descriptions of how this language is used to help students learn mathematics see Dautermann (1992) and Dubinsky (1995)

4. Investigations of derivative rules.

Students were given several activities to investigate and attempt to "discover" as many of the derivative rules using their symbolic computer system as they could manage. In addition, they investigated the derivatives of exponential functions and were asked to predict a number a so that the function f given by $f(x) = a^x$ was the same function as f by looking at graphs of f and $D(f)$ to get at the derivative of the natural exponential function, exp, represented by the expression $e^x = \exp(x)$. In addition, students investigated the difference quotient of the exponential function f with base a at the point 0 to find a number a to four decimal places, with $2 < a < 3$, so that the limit of the difference quotient is (close to) 1.

This activity relates to points 6 and 7 of our initial genetic decomposition.

5. Investigations of various applications.

Students were given various activities which involved applications related to slopes of tangents, graphical interpretations of derivatives, and instantaneous rates of change.

These activities relate to points 3a and 3b through 8 of our initial genetic decomposition.

Next, we discuss the extent of the understanding that we observed and indicate some possible comparisons between students from the C^4L and traditional calculus courses.

3 Results

In this section, we use excerpts from the interviews to describe various mental constructions that the students appeared to make or have difficulty in making. Then we present two kinds of comparisons of performance between C^4L students and students from traditional courses.

3.1 Mental constructions

In this section, we present our analyses of students' responses to Question 6, where they were given a graph of a function and a tangent line. The mathematical issues raised by this situation are: an understanding of a function represented by its graph, relating the slope of the tangent to the derivative, interpreting the tangent line, and determining the slope of a line. We will try to explain some of the students' varying degrees of success or failure in dealing with these issues in terms of the mental constructions of the APOS theoretical perspective.

This discussion will also refer to Question 7 in order to clarify our interpretation of a student's understanding. While we do not present an in-depth analysis of Question 7 at this time, the performance of students on this question gives us some information, in a different context, about their ability to deal with a function without an analytic expression for it.

An understanding of a function represented by its graph

The responses of the students to Question 6 (and to some extent, Question 7) present a fairly clear picture of a struggle to construct a process conception of function as the student tries to relate a given graph to an understanding of some function which has not otherwise been presented. The question of a student indicating a process conception of function by understanding how to use its graph to evaluate points has been considered previously (Breidenbach et al., 1992; Dubinsky & Harel, 1992). Our observations appear to be in line with these earlier studies which suggest that, although working with the graph alone may not help students much in developing a process conception of function, writing appropriate programs to implement these functions can make a difference.

We start with some examples of students who seem to have a process conception of function that they apply to this problem and succeed in solving it. They demonstrate their process conception of function as they interpret the notation $f(a)$, where a is a number, for the interviewer.[6] We begin with a comment from the student Hank who may be making the process connection between function and graph, provided "this function" is interpreted to mean the displayed graph.

Hank: I know, $f(5)$ is asking for the corresponding y value that would correspond with every single x value along this function and in this case it's 4.

We now examine a portion of the interview with the student Nate who also appears to make the connection.

I: Okay, so now when you say $f(5)$ is the point $(5,4)$, um, what does that mean exactly?

Nate: Um, let's see, $y = f(x)$ ...

I: Mmm-hmmm.

Nate: Um ... and since 5 is your x coordinate, if you look at the graph, I can go 5 in the x, and, the function, I don't know. I had it figured out. Well if you go 5 in the x, you only place the function, the y coordinate at, on the function at 5 on the x at 4 on the y.

I: Okay.

Nate: So, $f(5)$, 5 being the x, 4 has to be the y because it follows the curve.

I: Okay, okay.

In some cases, the student went to considerable lengths to find an expression in order to evaluate $f(5)$ by plugging in 5. For example, as illustrated in the next excerpt, some students used the coordinates of the point $(5,4)$ and calculated the slope from information on the graph to compute the equation of the line in the form $y = mx + b$ and then substituted 5 for x to get the correct answer 4.

I: Okay, okay. Let's move on to number 6. Did you know we were on 6 already?

Kurt: No.

[6] We use the notation I: for the interviewer in all excerpts even though the actual interviewing was done by several different members of *RUMEC*.

I: Okay, um, let's see. Well, I'll let you read that.

Kurt: Okay. [Long pause] ... [mumbling] ... [long pause] ... hmmm ... [pause] ... okay.

I: Okay, now explain to me what you've done there.

Kurt: I just found the, uh, equation for the line L.

I: Okay.

Kurt: And then, um ... so I had like, an equation in the form of $y = mx + b$.

I: Mmm-hmmm.

Kurt: And I, uh, plugged in the slope m and then for the, for the value at 5 I just plugged in 5 for the, for the x value.

I: Okay, now is that the value of f at 5?

Kurt: Yeah, f at 5. Right?

An even more convoluted approach was to calculate the slope of the line from the graph, integrate that value to get a linear expression and evaluate it at 5 to get the value of the function. This kind of reasoning was found only in the traditional group of students. In the following illustration of this triumph of calculation over concept, the student does begin to see that the answer could have been obtained more directly.

Lara: Ok, and just take the integral... Is it just that? Or is, I mean, is there more to it?

I: Well, does that look right with the graph?

Lara: Not really. Well,...

I: If you integrate $\frac{2}{5}$, but you don't have any limits of integration, ...

Lara: Right, what do you do... you get $\frac{2}{5}x$...

I: Is that all?

Lara: Um, dx... No,... plus c. I always forget that.

I: Ok, ...

Lara: Ok, now $f(x)$... so it's going to be $2 + c$.

I: Ok, so is this $f(x)$ for every x?

Lara: No, just for that one point.

I: So, just for 5, right?

Lara: Yeah... or at least for that one point... so then... Hmmm these two don't correlate, do they. They just look similar.

I: Yeah.

Lara: Now, I guess for that point we know, ... can equal 4... so this is it.

I: So we have $f(5) = 4$ and $f'(5)$ is...

Lara: is $\frac{2}{5}$.

We cannot say if Kurt and Lara are completely limited to an action conception of function, but it does seem clear that in dealing with this particular graphical situation they are not able to use any process conception to solve the problem. Their strong need to find or even construct an expression in which to substitute suggests that they may not be very far advanced beyond an action conception of function. The next example further illustrates this phenomena, again suggesting that, in a graphical context, the student may still be at the action level in her conception of function.

Alice: [Pauses while working on problem.]

I: Okay. Now, what did you do here?

Alice: Okay. Well, this is implicit differentiation.

I: Okay.

Alice: But I can't remember how to do it.

I: Okay

Alice: Umm ... I know that 1/2 would be, that's going to be the coefficient and this is going to be the x that I substitute into.

I: Uh-huh.

Alice: So this is, this is really a 5.

I: Okay. Now going back to this, uh, what is this equation from?

Alice: The equation of the line.

I: Okay.

Alice: And then y is $f(x)$, so I substitute 5 for ...

I: So you were taking a line to be $f(x)$?

Alice: Right.

I: Okay. What, what ...

Alice: Oh, you want the function.

I: Yeah, the curve there ...

Alice: Oh.

I: ... would be $f(x)$.

Alice: Well ... do I need to know the equation of the curve?

I: Umm ... no. There's enough information there, uh, to find $f(5)$ and $f'(5)$.

Alice: Okay. Well ... well ... well, you know x, so y has to be 0.

I: Okay. If the curve goes through this point, when x is 5 ...

Alice: I don't know how to find the equation.

Finally, some of the responses showed an absence of success of any kind, even in the presence of strong prompts from the interviewer. In some cases, the student is quite explicit about needing to find an expression before being able to proceed.

I: OK, Let me give you another problem, it's a little different, How about this one? (pause)

Irma: Suppose that the line is tangent to the graph at the point $(5, 4)$. Find the function at 5 and the derivative at 5. Great... Ha (laughs) Um, the, the tangent, the point tangent is the limit of this line z, and the limit is the derivative ...

I: Of it?

Irma: I don't know, it's all linked together and I don't , it's blocking out right now how...

I: How about $f(5)$, you called it the function of 5. Any ideas about that?

Irma: Well, this line is defined as $y = f(x)$.

I: Uh-huh.

Irma: and, so, whatever this curve is, this would substituted $f(5)$.

I: Uh-huh.

Irma: Ha (laughs) I don't have any idea, I don't know, if that's all that's given. I don't know, I'm feeling really stupid right now, but….

In the above excerpt, the student seems to be aware that f represents the function and might even understand that $f(5)$ represents the result of the process of the function applied to 5, but she is not able to make any connection between this and the information provided by the graph.

Relating the slope of the tangent to the derivative

We come now to what is, from the mathematical point of view, a key issue in a graphical understanding the derivative: the relationship between the derivative of a function at a point and the slope of the line tangent to the graph of the function at that point. This forms a foundation for understanding the derivative as a function which, among other things, gives for each point in the domain of the derivative the corresponding value of the slope.

Our data permits us to look at three aspects of this general question: Does the student understand that, for a point a in the domain of a function f, the value of $f'(a)$ is the slope of the line tangent to the graph of f at the point $(a, f(a))$? In the absence of an expression for the function, is the student able to think about and work with the derivative using only graphical information? Finally, to what extent can the student use this relationship between derivative and slope to obtain information about the graph of the function?

Regarding the first two questions, the students in this study appeared to fall into three categories: a reasonable understanding; lack of understanding but with some indications of progress, either present or appearing to be imminent; and serious misunderstandings. Once again, we will see that the main difficulty that prevents success appears to be the student's lack of a process conception of function. It seems that a student might have a process concept of function when thinking about a specific function (with or without expression), but that this can break down when the problem requires consideration of the derivative — even at a single point.

Let's consider first the category of students who seemed to have constructed an understanding of the relation between derivative and slope. In the following excerpt, Elmo provides a clear, straightforward explanation, working with functional notation without the slightest reference to an expression for the function.

I: OK. Let's move to the problem number six.

Elmo: f at the point … line L is tangent to a the function f at the point $(5, 4)$. OK. $f(5)$ is simply whatever your function is, where your function is on the y coordinate at x equals 5 which is 4. OK. $f'(5)$ … OK … now. $f'(5)$ gives you the slope of, of the line tangent at x equals 5 which would be … (mumbling) … $\frac{2}{5}$.

I: How did you find that?

Elmo: Just find the slope of this line using this two points, using $(0, 2)$ and $(5, 4)$.

In some cases, the student needed some prompting before arriving at essentially the same place. There were also some students whose understanding, perhaps recently constructed, was somewhat tentative. These individuals expressed a desire for an expression but, when one was not available, went on and solved the problem without it. In the following excerpt, Steve has no difficulty interpreting the value of the function graphically, but expresses a need for an equation fairly strongly. Then, however, he goes on to compute the slope of the tangent line and finally realizes that it is equal to the derivative. He does all of this without any prompting.

Steve: What am I doing? OK. Umm ... I'm making an equation for the ... umm ... let see... OK,. I'm making the equation for the line that is tangent to that point since you want to deal with only the quantities at that point. Umm ... for $f(5)$ it's already known since it's given on the graph. $f(5)$ would be ... umm ... that quantity would be 4. You got to find out the slope and to do that I only know how to that with an equation and not looking at your graph, it would be estimating. So, value of the ... let see ... the slope of the line at the point tangent is equal to that at the other point tangent. You put this into the equation and do the algebra.

I: How did you get this $\frac{2}{5}$?

Steve: 2 over 5, that is the slope. Simple, it's change in y over the change in x. For my ... y is 2 and x is 0. Since you know the slope then I guess you have $f'(5)$. Then $f(5)$ is 4 and $f'(5)$ is $\frac{2}{5}$.

There were students who may have been in transition towards making the connection between derivative and slope of tangent, as indicated by the presence of comments that suggested, but did not state explicitly, that the slope of the tangent was equal to the derivative. Carl is one example. Again, he has no difficulty with understanding the value of the function graphically.

I: Okay, [Carl], so what do you think about problem six.

Carl: Alright, well, $f(5)$, since it's on the graph and we have a point, um, at $x = 5$, we can just say that $f(5)$ is equal to 4.

I: Okay.

Carl: Okay, now if we, if we just had a typical graph we could find the derivative just by graphically drawing the derivative, um, but since we just, just by looking at it we can't really tell exactly where that's going to fall.

I: I believe you've got, it says for us that L is tangent.

Carl: Right, you've got the tangent. So what we can do is we can figure out the slope, um, of this curve at that point, um, just by delta y or delta x, the slope formula, um, and that's just going to be equal to, uh, let's see, four minus two over, uh, five minus zero, right, uh, which is going to be, uh, two-fifths for a slope, plug it into the point, point-slope formula, um, $y - y_0$, in this case it's 4 is equal to your slope times, uh, $x - x_0$, in this case is 4, whoops it's 5, um, so the equation at that point is going to be, uh, two-fifths x minus, uh, two plus four is six.

I: Mmm-hmmm.

Carl: Now if we take the derivative we can just say for this point, uh, let's, let's just look at this point only and forget everything else. Um, y prime as we just defined, by the point-slope formula, uh the derivative is going to be two-fifths. So, since we're, since this is the derivative of $f(x)$, um, or the derivative of y at that specific point, um, and that's a function, you can just say that, uh, $f'(5)$ would equal two-fifths I believe. Um, since we're, I mean, for that point.

Among the students who showed little progress we see, once again, that the search for an expression which can be differentiated by applying rules is the overpowering idea. Some students, such as Larry, found $f'(5)$ by first computing the equation of the tangent line (computing the slope in the process) and then differentiating this expression! Note that he states the correct formula for finding the slope, but then incorrectly computes it.

Larry: (Long pause) OK, uh, I guess what I could do is, um, I think I'd want to figure out what the equation is for the line first, and I can express that by $y = mx + b$. And, so, to find m as the slope, I would, um, use the slope formula, which is, um, $y_2 - y_1$ over x_2, excuse me, over $x_2 - x_1$.

I: OK.

Larry: ...and, we could use the points $(5, 4)$ as, yeah, $(5, 4)$ and $(0, 2)$, and it doesn't matter which order you use them in, just as long as you are consistent. And so I could use, um, 4 minus 0 over 5 minus 2, which is equal to 4 over 3 or, excuse me, four thirds is the slope. And then, um... putting that back into the equation and substituting, um, m for $\frac{4}{3}$, or $\frac{4}{3}$ in for m.

I: OK.

Larry: And, um, then... I think what I'd want to do is to use one of these points, for instance, $(5, 4)$, and plug that in for y and x in this. I don't want to do that. I've got to do something with b. But I'm not quite sure what I'd want to do with b. Actually, I don't know how to carry this out. But, what I — somehow I've got to get the equation for the line.

I: OK. So let's just pretend that b was whatever that number is that you're after. Let's say you had found that already. Because you say you're not sure how to come up with that b right now. Is that it?

Larry: Yeah.

I: OK. So let's say you already have that. What would you do with it next? $f(5)$ and $f'(5)$?

Larry: What would I do? Well, if I wanted $f(5)$, all I'd have to do is substitute in 5, um, into the equation that I have. And then for $f'(5)$, I would take the derivative of the original equation and then substitute in 5 for that.

Larry then proceeds to make this last calculation, differentiates the linear expression, notes it is just a constant so there is no place to substitute 5, and gives the constant as the answer. The student appears unaware of the equivalence between solving the given problem (finding the derivative of f) and the problem he solved (finding the derivative of the expression representing the linear function).

Question 7, which required the construction of a graph from information about the derivative of the function tended to confirm these categories. With a few exceptions, the students in the first category did well on the graphs and the students in the third category did poorly. The small number of students in the middle category showed no pattern in their attempts to construct the graph.

Interpreting the tangent line L

We have discussed above how the students work with a function and its derivative given only graphical information. In dealing with this situation, the students had to make sense of the line tangent to the function f at the point $(5, 4)$. The mathematical interpretation of the tangent line in this situation is that it is an auxiliary feature for the curve. As a function, the tangent line agrees with the original function at the point of tangency and its slope (i.e., the derivative of this linear function) is the derivative of the original function at that point. In some cases, the student identified the function underlying the tangent line with the original function, and in others they identified it with the derivative of that function.

The few students who made use of the expression for a linear function (with graph L) in their work indicated that they are not merely substituting a known expression for an unknown function. Dale seems to understand that at the point $(5, 4)$ the line and function are virtually the same, so it makes sense to take the derivative of the expression for L. Note that the interviewer is the one who states this. We interpret this as a summing up of Dale's intent, which may have been communicated non-verbally through a hand gesture or some other similar cue.

I: Okay. I'm not sure I followed that. I agree with the answer but how did it, how did you get there?

Dale: Okay. The, okay, the line and the function ...

I: Uh-huh.

Dale: ... cross at five.

I: Uh-huh.

Dale: So the equation at this point for the line, for the line and the function here at this kind of sine little thing ...

I: Uh-huh.

Dale: ... are going to be equal. So you find the equation of the line at this point and take the derivative of that.

I: Oh, okay, so they have the same derivative at that point.

Dale: Yes.

Other students made use of the expression for L without indicating any understanding that it was a linear approximation of f at the point. Here Larry discusses the second part of Question 6, continued from Section 3.1 where he found an equation for the line, L. He is unable (or unwilling) to determine a value for b in his slope-intercept form of the line. The interviewer provides the value $b = 2$ and asks him to explain how he would finish the problem. Apparently, he sees no difference between f and L.

Larry: Well, I would say... um, OK, so then I would have $y = \frac{4}{3}x + 2$. And then I would want $f(5)$, so I'd substitute that in. Say, $f(5)$ is equal to $\frac{4}{3}$ times $5 + 2$, and so, we'll say $f(5)$ equal to... $\frac{20}{3}$ plus, um, $\frac{6}{3}$ ... be $\frac{26}{3}$. And... um, taking this up here, uh, for $f'(5)$ we would, um, have to take the derivative. And that would be... just, hmm...

I: Just what?

Larry: Well, I'm trying to think here, if I take the derivative of that, that would just give me $\frac{4}{3}$. The x would — I wouldn't be able to substitute in the 5. And so, I would say it would be $\frac{4}{3}$.

I: OK, does it bother you that you can't substitute that 5 in? Is $\frac{4}{3}$ a good answer?

Larry: Usually when you have to — when you've got a number there, you need to plug it in somewhere. But, taking the derivative of that, and I don't anything, I mean, I don't have any, um, unknowns left, which would just be $\frac{4}{3}$. I'd say that's what it would have to be.

In the following excerpt, Uriah not only states explicitly that the line is the derivative, but he also indicates that he is talking about a function by his reference at the end "to plug it in" to get the value of the derivative at 5.

I: Okay, okay. Um, let's move on then ... to number six.

Uriah: (Mumbling) ... $f(5)$ ... here's a L, L is tangent to the graph of the function f ... okay, then we need to find $f(5)$ ... $f(5)$ would be, this would be 4, it's got to be.

I: Okay.

Uriah: $f(5) = 4$, and the equation of this line is, $y = mx + b$, so the rise over run equals ... $\frac{1}{2}$ and $\frac{1}{2}x + 2$ so, you just want f derivative at 5, which is this line ... at 5, so I get it equals $\frac{5}{2} + 2$, $\frac{9}{2}$.

I: Okay.

Uriah: I guess that's right.

I: Okay, now explain this part to me a little more.

Uriah: Well, when you have a curve, the derivative is the line tangent...

I: Okay.

Uriah: ...and so, uh, if I find the equation of this line, I figured I'd find the equation of the derivative...

I: Okay.

Uriah: ...according to the coordinates.

I: Uh-huh.

Uriah: And then just plug in whatever value.

In light of the confusion between the line L and the graph of the original function, it might be useful to summarize the various methods the students in this study used to compute the slope of L. In addition to noting that some students did not compute the slope, we observed the following two methods that were used to compute slope: (i) computing rise over run from two points and (ii) finding an expression for L and differentiate it to find the slope.

3.2 Performances

Here we describe the performance of the students in making the proposed mental constructions and developing the desired mathematical understanding. We include some information that breaks the population into two groups: C^4L and traditional, but we do not analyze these numbers. From a very informal point of view, the reader can see that, in general, the percentage of the 17 students from the C^4L course who seem to succeed, or avoid difficulty is higher than the corresponding percentage of the 24 students from the traditional courses.

We consider separately graphical understanding of a function and graphical understanding of its derivative.

An understanding of a function represented by its graph

All of the 17 students from the C^4L course demonstrated a process conception of function, both in their apparent understanding of the $f(x)$ notation and in their ability to use the graph in the absence of an expression. Of these, one student needed a very small amount of prompting on the $f(x)$ notation and two expressed a strong desire to have an expression, but when it became clear that none was available, were able to go on working without it. None of the C^4L students attempted to use an expression for the line to evaluate the function.

Of the 24 students from the traditional classes, 16 seemed to understand the $f(x)$ notation at a level similar to that of the C^4L students and also could use the graph to solve the problem. In addition, there was one student who appeared to understand the $f(x)$ notation and one who appeared to be able to use the graph to solve the problem, but did not understand the $f(x)$ notation. Included in these 18 were one student who needed a very small amount of prompting for the $f(x)$ notation and three who definitely needed prompting. Also included in this 18 were four students who expressed a strong desire to have an expression, but when it became clear that none was available, were able to go on working without it.

The remaining six students from the traditional classes were unable to work on this problem without having an expression. They calculated an expression for the line L by either using analytic geometry or integrating the slope of the line, and then evaluated it at the x-value to find the y-value. Initially, there were six students who used analytic geometry and five who integrated. Some members of these two groups eventually improved their responses and are counted in the group of 18, which is why the total adds up to more than 24.

Relating the slope of the tangent to the derivative

A group of $22(11, 11)$[7] students displayed a reasonable understanding of the relation between the slope of the tangent and the derivative. An additional group of $3(1, 2)$ students were close to this level, needing only a little prompting. There were $3(2, 1)$ students who were in the middle and $13(10, 3)$ who did not display very much understanding and even appeared to have serious misconceptions. A total of six students, $1(0, 1)$ from the middle group and $5(5, 0)$ from the group without understanding, computed the equation of the line and took its derivative.

All of the 25 students in the first two groups were successful in using information about the derivative to determine intervals of monotonicity for the graph in Question 7 and all but $5(3, 2)$ of them indicated in written work and/or discussion that they realized that the slope becomes infinite, or a vertical tangent line is obtained as the derivative increases without bound. On the other hand, of the 13 students who did not display much understanding, $5(3, 2)$ succeeded in identifying intervals of monotonicity and knew that the slope becomes infinite and an additional $2(1, 1)$ were successful only with the first of these. Finally, of the 3 students in the middle, $2(1, 1)$ succeeded on both tasks and the other $1(0, 1)$ was unsuccessful on both.

As seen in the previous section, most of the students correctly interpreted the tangent line in the diagram. Of the students who interpreted the line differently, $2(2, 0)$ students indicated they saw L as an approximation of f; $4(3, 1)$ others took the derivative of an expression for L;

[7] Our notation here gives the number of those in the traditional sections first, followed by the number from the C^4L course. See the tables on p. 25.

and $7(4,3)$ students assigned the expression for L to f'. The remaining $3(2,1)$ students from the middle group or the group without understanding were not able to say anything meaningful about the situation.

A total of $26(11,15)$ students used the two-point formula to compute the slope without difficulty and an additional $5(5,0)$ did this with some prompting and/or made some errors. There were $6(5,1)$ students who found an expression for L and differentiated and $4(3,1)$ who were unable to compute the slope.

3.3 Comparison of graphical understandings by students from traditional and C^4L courses

Comparisons of results

Now we focus more on mathematical performance and a direct comparison of the students from the C^4L course with the students from the traditional courses. As we indicate below, this comparison is flawed in many ways and great care should be taken in drawing any conclusions from it. We do feel, however, that because the numerical differences are so striking, it is worth presenting the information for what it may be worth.

We compare how the students appeared to perform, in the opinion of the four researchers (with differences negotiated) on the 8 questions the described in Section 2.3 on Methodology, p. 8. As we noted there, for our purposes in this report, we simplified the coding to the following three point system by merging categories in some questions and/or re-interpreting others.

3. The student appeared to understand completely.

2. The student appeared to have the main ideas, possibly only after some prompting by the interviewer and/or some specific errors were made.

1. The student displayed little or no understanding.

In the following tables, we present for each of the two groups of four questions used by the researchers, the quadruple of scores obtained by the students and the number of students who obtained each quadruple, in each group — the C^4L and traditional (TRAD) students.

Graphical Understanding:

Of a Function

F1	F2	F3	F4	TRAD (24)	C^4L (17)
3	3	3	3	6	9
3	3	3	2	4	5
3	3	2	3	0	1
2	3	3	3	1	1
3	3	3	1	1	1
3	3	2	2	1	0
2	3	3	1	1	0
3	3	2	1	1	0
2	3	3	1	1	0
1	3	3	1	1	0
2	1	3	2	1	0
1	1	3	2	1	0
1	1	3	1	1	0
1	1	1	2	2	0
1	1	2	1	2	0

Of the Derivative of a Function

D1	D2	D3	D4	TRAD (24)	C^4L (17)
3	3	3	3	9	9
2	3	3	3	0	3
3	3	1	3	1	3
1	3	3	3	0	2
2	1	3	3	1	0
2	3	1	3	1	0
1	1	3	3	4	0
3	3	1	1	1	0
1	1	3	1	1	0
1	1	1	3	1	0
2	1	1	1	1	0
1	1	1	1	4	0

We cannot say with any degree of certainty that these codes represent what the students have learned about graphical representations of a function and its derivative. Learning is much too complex to be reducible to integers in a small range. Nevertheless, we feel that it is worth looking at these codes for two reasons. One is that they may give us some indication or approximation to the learning that took place. The other is that these are the kinds of numbers that are usually studied in standard assessments of learning, and we feel that the reader might benefit from seeing what such numbers look like for these two groups of students.

It may be worthwhile then to ask what story this data tells. The data seems to be very clear about how much the students learned on two counts: how this learning appears to vary between the two groups and how it varies between the function and its derivative.

On both counts, the data appears to be very informative. If we take as a completely satisfactory outcome that a student was assigned the highest code on three of the four questions then, on the one hand, we see that *all* of the students from C^4L calculus achieved this level on both of the two sets of questions. In fact, on each set, 9 of the 17 students (not always the same 9) got the highest code on all four questions. On the function question, an additional 7 got the next highest code on the one question where they did not get the highest. On the derivative question, an additional 3 got the next highest code on the one question where they did not get the highest.

On the other hand, only half of the 24 students who took the traditional calculus courses received the highest code on all but one of the function questions and only 10 were at this level on the derivative questions.

Moreover, amongst the 24 students who took the traditional courses there were 5 who got the lowest code on all but one of the function questions and 7 who did as poorly on the derivative questions.

Thus we might summarize the story that this data tells as follows. Nearly half of the students who took traditional calculus courses were assigned completely satisfactory codes, about a fourth appeared to do very poorly, and the rest were in between these two extremes. The students who took C^4L calculus all received completely satisfactory codes.

Turning now to a comparison between the function and the derivative questions, we see that both groups of students show a small drop in performance in going from the former to the latter. For the students from the traditional calculus courses, the number of students who were completely satisfactory dropped from 12 to 10 and there was a corresponding increase in the number of students who did poorly, the number in the middle staying the same. For the C^4L students the drop is indicated by an increase from 1 to 5 in the number of students who received the lowest code on one question (none of these students received the lowest code on more than one question).

Comparisons of students in the two groups

The results we have just described suggest that the C^4L students in this study performed at a much higher level on the interview questions than the traditional students. This raises the question of possible differences between the two groups of students. Is there any data that might suggest that the difference is due to factors other than the difference in the two courses? For example, we might ask if the students who took the C^4L courses were stronger students who might do better no matter what course they took.

The data we have on this question is far from conclusive. There are a vast number of factors that could have caused the difference in the codes assigned, including the possibility that the C^4L method is better for some students and these are the ones who remained in the course after seeing what it was about or switched to it from the traditional course after hearing about it from colleagues. This factor remains a possibility even if, as is the case here, very few of the students actually did switch from one method to the other.

It seems impossible to control for all such factors. We can only present one small piece of data and even that is not complete. The registrar at the school in which this study took place computes what is called a *predicted grade point average* (PGPA) for most students entering the university. This is based on data such as high school grades, SAT/ACT scores, and similar factors. It is calculated before the student begins her or his studies and it has turned out to be fairly accurate in predicting the actual grade point average which the student obtains. It is a number ranging from 0 to 400 where, for example, 265 corresponds to 2.65 on the standard four point grade scale (4 corresponds to A, 3 to B, 2 to C, 1 to D and 0 to F).

In order to protect the identity of individual students, we submitted a request for student PGPAs by listing the students in three groups. We were able to identify 7 of the 24 students from the traditional calculus course whose numerical codes were uniformly low on all questions we considered. We obtained PGPA data for a total of 30 of the 41 students in this study; 15 of the 17 C^4L students, 5 of the 7 "low" students, and 10 of the 17 remaining from the traditional sections.

This low group is important because, if you drop the 7 students with the lowest codes from the students who took the traditional course, then the differences between the two groups are fairly small. Thus we might have a situation in which all of the weak students were in the traditional courses. Our data suggests that this was not the case.

Thus we report three sets of data:

R1 5 students who took the traditional courses and received the lowest codes.

R2 10 other students who took the traditional courses.

A 15 students who took the C^4L courses.

PGPA Scores

	R1	**R2**	**A**
	265	211	210
	280	254	212
	297	262	230
	298	276	232
	322	301	244
		301	250
		316	266
		317	281
		321	282
		327	283
			290
			303
			304
			319
			340
Average	292.4	288.6	269.7

Of course we cannot conclude anything from this data because it is incomplete. Nevertheless, the reasons for the missing scores have to do only with technical problems of identifying students without violating confidentiality and nothing to do with mathematical ability. Thus, there is no reason to believe that adding PGPAs for the other 11 students would make a difference in the picture that is presented.

This picture is very clear. The PGPAs of the students who took the traditional courses are, as a group, a little bit higher than those who took the C^4L courses. In fact, the students who received the lowest codes have the highest PGPAs. However, the differences are not statistically significant.[8] The one thing that is certain is that these numbers do not suggest that the students who took the C^4L courses were better students than those who took the traditional courses, or that the lower codes were due to the presence of an identifiable subgroup whom we could expect to do poorly under almost any circumstances.

4 Discussion — a revised genetic decomposition

We now present a revision of our initial genetic decomposition of the graphical understanding of a function and its derivative which guided the instruction of the C^4L course and the subsequent interviews which resulted in this data. The revision results from attempting to use the initial genetic decomposition to explain the mental constructions seen in the interview data presented above. We focus our revision on the interplay of the concept of function and the necessity for an expression that some students exhibit.

Our initial genetic decomposition, presented in Section 2.4, describes parallel constructions of the derivative in its analytic definition and its graphical interpretation. The mental constructions we observed in this study (see Section 3) are for the most part contained in it. However, certain aspects of the student's concept of function need to be emphasized, as well as creating links between the paths.

Prerequisite knowledge The initial genetic decomposition states as prerequisite knowledge that students will have a working knowledge of and an understanding of the following: the graphical representation of points, the concept of slope of a line, and the concept of function. As seen in Section 3.2, two aspects of the function concept need to be explicitly addressed. Excerpts from the interviews suggest that not all students understand how the point (x, y) should be interpreted when y is given by $f(x)$. Other excerpts point out how difficult it is for some students to overcome the necessity for a formula.

Specifically, we revise the initial genetic decomposition to state as prerequisite knowledge:

[8] Comparing the two groups of 15 scores in a one-tail test gives $p = 0.067$, the two-tail test results in $p = 0.134$. Significance is not obtained unless $p < 0.05$.

A. Graphical representations of mathematical objects

   i. Graphical representation of a point

   ii. Graphical representation of a line including the concept of slope

B. Coordinating representations of points with a function

   i. Graphical interpretation of (x, y) when y is given by $f(x)$

      Even though a student may exhibit a process conception of function in situations where functions are represented by expressions, equations or sets of ordered pairs, this same student could be at an action level when dealing with the graphical situation. An action conception is indicated when a student has a need for a formula for the function. This necessity may cause the student to find an expression for some other related function to substitute for the given function.

   ii. Overcoming the need to have a formula for the function

      In the graphical situation, the process of interpreting a point on the graph can be seen as "the corresponding y value that would correspond with every single x value along this function," or "5 being the x, 4 has to be the y because it follows the curve" from the interview excerpts in Section 3.2.

Graphical interpretation of the derivative The mental constructions discussed in Section 3.1 are accounted for in the initial genetic decomposition for the most part. For example, on p. 19, Steve seems to be on the verge of constructing point 5 of the initial genetic decomposition (see pp. 10–11), having demonstrated points 3a and 3b. Similarly, on p. 20, Larry's responses suggest that he has reached point 4 — possibly without making any graphical interpretation. Our revisions here focus on the need to tie together the two paths as the points are constructed. This may help overcome the need to have an expression to differentiate.

For example, one type of construction that we observed was the use of integration as students tried to make sense of the situation. Some students integrated the value of the slope of the tangent line at the given point, as a constant, to find an expression (linear) which they used as if it represented the original function. These students exhibited a very strong need for an expression representing the function to differentiate and then evaluate, rather than being able to work with the local data and the interpretation of the derivative as the slope of a tangent line at the point. In particular, these errors indicate that the student is not constructing the derivative function from the derivative at a point. Hence, it makes sense to the student to take the derivative at a point and integrate that to find the original function. Focusing on building the derivative function should address the difficulties students have with interpreting the tangent line as presented in Question 6.

We replace points 5–7 in the initial genetic decomposition (see p. 11) with the following:

C. Graphical interpretation of the derivative at a point

   i. Overcoming the need to differentiate some formula

      Students indicating a need to take a derivative have a pre-action conception of the derivative graphically. That is, they demonstrate no knowledge of this interpretation. However it is also possible that such understanding exists but is not evoked by the problem situation.

   ii. Coordinate with A to see $f'(a)$ as the slope of a tangent line

      As an action, the tangent line is identified at the point, and its slope is computed. There are (at least) two ways for this action to be built. The student may simply have memorized a rule: the derivative of a function at a point is the slope of the tangent line to the graph at that point. A richer construction comes from building the tangent as the limit of chords of the curve $y = f(x)$, as in points 1a–3a in the initial genetic decomposition. From this action the student will construct the process of the derivative function. A student demonstrates this as an action when he or she focuses on one point in the domain at a time.

   iii. Coordinate several interpretations of $f'(a)$

      The student brings together the ideas of limit of difference quotient, average velocity, marginal cost, etc. and is able to move between interpretations.

D. Graphical interpretation of the derivative as a function

   i. Seeing the derivative as the function, $x \to$ slope at $(x, f(x))$

      In the graphical situation, this is the process conception of the derivative. This construction may be concurrent with, or in addition to, the student's understanding that the derivative of a function is itself a function.

   ii. Identifying f' with the line tangent at a point

      This misconception may arise as the student sees that the derivative is a function but has not coordinated the graphical interpretation with the analytic. The line is seen as the graph of a second function which to the student must be the derivative.

Using the concept of derivative In light of this study, we desire to make point 8 of the initial genetic decomposition more explicit on the issue of using (reversing) the notion of derivative to obtain the graph of the original function. This reversal indicates a clear process conception of the graphical interpretation of derivative which may be the de-encapsulation of an object conception.

E. Several coordinations to get the graph of f

   i. Graphical interpretation of $f(x)$, for a single x

   ii. Interpretation of $f'(x)$ for a single x as the slope

   iii. Process of x moving through an interval

      a. monotonicity of the function and sign of the derivative

      b. infinite slope (vertical tangent) and infinite derivative

      c. concavity of the function and sign of the second derivative

   iv. Drawing a complete, or fully representative graph

A revised genetic decomposition

Below we give a summary of the main points of our revised genetic decomposition (compare to our initial genetic decomposition on pp. 10–11).

Prerequisite knowledge.

A. Graphical representations of mathematical objects

   i. Graphical representation of a point

   ii. Graphical representation of a line including the concept of slope

B. Coordinating representations of points with a function

   i. Graphical interpretation of (x, y) when y is given by $f(x)$

   ii. Overcoming the need to have a formula for the function

Graphical and analytical paths to the derivative.

1a. Graphical: The action of connecting two points on a curve to form a chord which is a portion of the secant line through the two points together with the action of computing the slope of the secant line through the two points.

1b. Analytical: The action of computing the average rate of change by computing the difference quotient at a point.

2a. Graphical: Interiorization of the actions in point 1a to a single process as the two points on the graph get "closer and closer" together.

2b. Analytical: Interiorization of the actions in point 1b to a single process as the difference in the time intervals get "smaller and smaller", i.e., as the length of the time intervals get "closer and closer" to zero.

3a. Graphical: Encapsulation of the process in point 2a to produce the tangent line as the limiting position of the secant lines and also produce the slope of the tangent line at a point on the graph of a function.

3b. Analytical: Encapsulation of the process in point 2b to produce the instantaneous rate of change of one variable with respect to another.

4. Interiorization of the processes in points 2a and 2b in general, to produce the definition of the derivative of a function at a point as a limit of a difference quotient at the point.

Graphical interpretation of the derivative.

C. Graphical interpretation of the derivative at a point

i. Overcoming the need to differentiate some formula

ii. Coordinate with point A to see $f'(a)$ as the slope of a tangent line

iii. Coordinate several interpretations of $f'(a)$

D. Graphical interpretation of the derivative as a function

i. Seeing the derivative as the function, $x \to$ slope at $(x, f(x))$

ii. Identifying f' with the line tangent at a point

Using the concept of derivative.

E. Several coordinations to get the graph of f

i. Graphical interpretation of $f(x)$, for a single x

ii. Interpretation of $f'(x)$ for a single x as the slope

iii. Process of x moving through an interval

a. monotonicity of the function and sign of the derivative

b. infinite slope (vertical tangent) and infinite derivative

c. concavity of the function and sign of the second derivative

iv. Drawing a complete, or fully representative graph

5 Conclusions, pedagogical suggestions and future studies

The action-process-object-schema, or APOS, theoretical framework served as a very useful tool for analyzing the students' graphical understanding of the derivative concept. In this study, we have tried to contribute to knowledge of how the concept of derivative can be learned by analyzing some points in the literature and by providing a genetic decomposition of the derivative concept. The results of this study suggest that the instructional treatment as given in C^4L calculus courses, or one similar to it, may indeed contribute to students acquiring a stronger conception of function and derivative (both as processes and as objects) than is possible in traditionally taught courses.

Based on the results of this study and other recent work on the function concept, we believe that our instructional treatment on functions, or one similar to it, is necessary for most college freshman entering calculus. This necessity cannot be over emphasized: pre-college mathematics courses must address the well-documented problem of students' inadequate preparation for calculus (and other higher mathematics courses) with respect to the function concept.

We should note that a genetic decomposition is a tool which we, as researchers, can use to make sense of data relating to a student's understanding of a concept. The two versions of a genetic decomposition for the derivative concept presented in this paper are grounded in the APOS theoretical framework, our own understanding of functions and derivatives, and our analysis of the interviews. In no sense are we claiming that our data "proves" anything about our genetic decompositions. We can only claim that the data illustrates and is consistent with the theoretical analysis. Through our analysis of in-depth clinical interviews, we have been able to further refine our understanding of students' conception of derivative.

Pedagogical suggestions. In general, the results and comparisons considered in Sections 3 and 4 suggest that the pedagogy used in the C^4L calculus courses relative to the concepts of function and derivative is reasonably effective in helping students develop a relatively strong process conception of function and a graphical understanding of derivative. For example, of the 17 C^4L calculus students in this study, all demonstrated a relatively strong process conception of function, both in their apparent understanding of the $f(x)$ notation and in their ability to use the graph and not rely on any expression to deal with Interview Questions 6 and 7. This contrasts with the observation that of the 24 traditional course calculus students in this study, six students were unable to work on Interview Question 6 without having an expression. Because of the performance of the students in the C^4L course, we do not see the need for any significant changes in pedagogy from the instructional treatment described in Section 2.5.

Future studies. We believe that the results given in this paper suggest that our instructional treatment on derivatives may be much more effective than that of a traditional calculus course. In future studies, we plan to continue this research effort by studying students whose calculus course uses the C^4L instructional treatment in an attempt to further refine and improve our genetic decomposition of the derivative concept and hence possibly improve our future instructional treatment. We also plan to investigate the lower levels (points 1–4) of our revised genetic decomposition of the derivative concept in future studies. In investigating the lower levels of our revised genetic decomposition, there are other questions we were not able to deal with in this study and must be addressed in future studies. In particular, we could not address the following questions with the data on which this study was based:

1. How does the student construct the relation of the derivative with the slope of the tangent line? Is this construction made simply from the statement that the value of the derivative at a point is the slope of the tangent, or is it from the limiting process on secant lines?
2. How does the student construct the concept of instantaneous rate of change of one variable with respect to another variable? Is this construction made simply from the statement that the value of the derivative at a point is the instantaneous rate of change of the function variable with respect to the independent variable, or is it from the limiting process on average rates?
3. In getting the intervals of monotonicity from data about the derivative of a function, what role is played, in the mind of the student, by the sign of the derivative? Is the result a remembered fact or is there any indication of an understanding of why the intervals of monotonicity are as they are, based on the rate of change of the function or something else?
4. In getting the intervals of concavity from data about the second derivative of a function, what role is played, in the mind of the student, by the sign of the second derivative? Is the result a remembered fact or is there any indication of an understanding of the relationship.

6 Acknowledgments

We would like to thank the members of *RUMEC* for their support, helpful discussions, comments and suggestions. In particular, we would like to thank Anne Brown, Julie Clark, David M. Mathews, Karen Thomas, and Georgia Tolias for their many insightful suggestions for improving this paper.

References

Amit, M., & Vinner, S. (1990). Some misconceptions in calculus: Anecdotes or the tip of the iceberg? In G. Booker, P. Cobb & T. N. de Mendicuti (Eds.), *Proceedings of the* 14^{th} *International Conference for the Psychology of Mathematics Education, 1* (pp. 3–10). Oaxtepec, Mexico: CINVESTAV.

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (in press). A framework for research and curriculum development in undergraduate mathematics education. *Research in Collegiate Mathematics Education*.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 26*, 135–164.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (in press). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of Mathematical Behavior*.

Dautermann, J. (1992). *Using ISETL 3.0: A language for learning mathematics*. St. Paul, MN: West Educational Publishing.

Dreyfus, T., & Eisenberg, T. (1983). The function concept in college students: Linearity, smoothness and periodicity. *Focus on Learning Problems in Mathematics, 5*, 119–132.

Dubinsky, E. (1995). ISETL: A programming language for learning mathematics. *Communications in Pure and Applied Mathematics, 48*, 1027–1051.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G. Harel and E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA notes 25* (pp. 85–106). Washington, DC: Mathematical Association of America.

Dubinsky, E., Schwingendorf, K. E., & Mathews, D. M. (1995). *Calculus, concepts and computers* (2nd ed.). New York, NY: McGraw-Hill.

Eisenberg, T. (1992). On the development of a sense for functions. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA notes 25* (pp. 153–174). Washington, DC: Mathematical Association of America.

Ferrini-Mundy, J. & Graham, K. G. (1994). Research in calculus learning: Understanding limits, derivatives, and integrals. In J. J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning, MAA notes 33* (pp. 31–45). Washington, DC: Mathematical Association of America.

Lauten, A. D., Graham, K., & Ferrini-Mundy, J. (1994). Student understanding of basic calculus concepts: Interaction with the graphics calculator. *Journal of Mathematical Behavior, 13*, 225–237.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*, 1–64.

Orton, A. (1983). Student's understanding of differentiation. *Educational Studies in Mathematics, 14*, 235–250.

Schoenfeld, A. H., Smith, J. P. III, & Arcavi, A. (1990). Learning — the microgenetic analysis of one student's understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology, 4* (pp. 55–175). Hillsdale, NJ: Erlbaum.

Schwingendorf, K., Hawks, J., & Beineke, J. (1992). Horizontal and vertical growth of the students' conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA notes 25* (pp. 133–149). Washington, DC: Mathematical Association of America.

Schwingendorf, K. E., Mathews,D. M. & Dubinsky, E. (1996). Calculus, concepts, computers and cooperative learning, C^4L: The Purdue calculus reform project. *Proceedings of the Seventh Annual International Conference on Technology in Collegiate Mathematics* (pp. 402–406). Reading, MA: Addison-Wesley.

Selden, J., Selden, A., & Mason, A. (1994). Even good calculus students can't solve nonroutine problems. In J. J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning, MAA notes 33* (pp. 31–45). Washington, DC: Mathematical Association of America.

Sfard, A. (1987). Two conceptions of mathematical notions, operational and structural. In A. Borbàs (Ed.), *Proceedings of the* 11^{th} *Annual Conference of the International Group for the Psychology of Mathematics Education* (pp. 162–169). Montreal: University of Montreal.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification — The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA notes 25* (pp. 59–84). Washington, DC: Mathematical Association of America.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495–511). New York: Macmillan.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. *Research in Collegiate Mathematics Education, 1*, 21–44.

Tufte, F. W. (1989). Revision in calculus instruction: Suggestions from cognitive science. In J. Malone, H. Burkhardt & C. Keitel (Eds.), *The mathematics curriculum: Towards the year 2000. Content, technology, teachers, dynamics* (pp. 155–161). Perth, Western Australia: The Science and Mathematics Education Centre, Curtin University of Technology.

Vinner, S. (1989). The avoidance of visual considerations in calculus students. *Focus on Learning Problems in Mathematics, 11*, 149–156.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*, 356–366.