COORDINATING VISUAL AND ANALYTIC STRATEGIES

A STUDY OF STUDENTS' UNDERSTANDING OF THE GROUP D_4

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Abstract

This study contributes to the ongoing discussion of visualization and analysis in mathematical thinking. Based on data gathered from clinical interviews with 32 students in their first Abstract Algebra course, we consider the tasks of listing the elements of the dihedral group D4 and finding the product of two such elements. These problems can be solved either using a "visual" approach of transforming a square or an "analytic" approach of multiplying permutations. Rather than clearly preferring either a visual or analytic strategy, most students in our study used some combination of these approaches. Our results suggest that the conventional analyzer/visualizer dichotomy may not be an appropriate classification scheme for describing learning processes or for designing instruction. We propose an alternative model, the Visualizer/Analyzer or VA model, that assumes visualization and analysis to be mutually dependent in mathematical problem solving, rather than unrelated opposites. Our model provides one description of how this mutual dependence might function. We end by considering how pedagogical approaches might be designed in consonance with this model to help students coordinate visual and analytic thinking.

"No soul thinks without a mental image."

(Aristotle)

Mathematics education research has long debated the relative presence and value of visual versus analytic elements of mathematical thinking. The issues at stake are broad and complex, even controversial. This study seeks to explore the interrelationship between visualization and analysis in terms of student learning.

We specifically examine here the thinking of undergraduate mathematics students when dealing with a situation in abstract algebra that has two very clearly different representations—one apparently visual and one apparently more analytic. We assume that understanding the ways students work with either or both of these representations may contribute to the ongoing discussion of the ways students may use visual or analytic approaches in learning advanced mathematical concepts.

Our specific situation deals with the dihedral group of order four, denoted D_4, and we will consider student thinking about two problems: list the elements of this group and calculate the products, according to the group operation, of pairs of elements. We chose to observe students working with these D_4 problems since (a) each of the interpretations described below represents roughly the same level of mathematical sophistication, (b) both processes are simple enough to be carried out quickly, and therefore manageable during a clinical interview, and (c) the situation itself is complex enough to bring out distinctions in the students' understanding.

The group D_4 can be modeled in two ways. The approach which we take to be highly related to visual thinking is expressed in terms of the symmetries of a square. In this view, the elements of the group are the four rotations of the square around its centroid—in 0, 90 180 and 270 degrees, together with 4 reflections or "flips" (across lines connecting the centers of the opposite sides and the two diagonals). The group operation between two symmetries consists of performing one symmetry on a square and then performing the other on the result. Using this approach, a mathematics student might employ a physical model of the square to achieve an understanding of its various rotations and flips. Figure 1 illustrates this method of calculating the product of two symmetries. The student performs a vertical flip followed by a 90-degree clockwise rotation to arrive at the reflection or flip across the diagonal with positive slope.

{Insert figure 1 here}

A second approach to D_4 which we take to be more representative of analytic thinking expresses the group in terms of permutations of four objects. The group operation in this case consists of applying a specific algorithm to multiply these objects and produce a permutation product. Thus a vertical flip of the square might be represented by the permutation $\begin{pmatrix}1234\\2143\end{pmatrix}$ and a 90-degree rotation by the permutation $\begin{pmatrix}1234\\2341\end{pmatrix}$. After multiplying these permutations we get $\begin{pmatrix}1234\\3214\end{pmatrix}$, which represents a flip of the square across its right diagonal. Our investigation attempts to tease out the relationship between these two ways of dealing with this situation and to explore how students may use them to learn about the dihedral group.

There seems to be little doubt that, as Piaget (1977) points out, "Some people are particularly visual, others mainly motor, auditory, etc." (p.684). This observation has to do with preferences for using one or another of the senses. Connecting it to pedagogical practices, however, has been complicated by variety of definitions of visualization as well as the tendency to see visual thinking as indicative of a student's overall approach to learning.

Attempts at defining visual thinking have been based variously on pictorial representations of objects, geometrical or graphical representations, questions of internal versus external representations, and intuition (see Lean & Clements, 1981; Pylyshyn, 1973; Webb, 1979; Zimmermann & Cunningham, 1991). Sometimes defined by what it is not, visual thinking has been set into such oppositions as visual versus verbal; visualized versus actual; visualization from memory versus perception in the present; visualization as spatial versus abstraction; thinking about images that are static versus images that change (i.e. move). In this paper we offer our own provisional definition of visual and analytic thinking as strategies and illustrate them using our observations of students.

Clements' (1982a, 1982b) summary of work in this area focuses on identifying personal traits of learners. He suggests that besides "visualizers" and "verbalizers" there appear to be "mixers," or individuals who "do not have a tendency one way or the other" (Clements, 1982b, p. 34). Clements' introduction of this middle category illustrates the difficulty of distinguishing any individual's particular preference for visual thinking in absolute terms. Anderson (1978) claimed that it is not possible to establish such a distinction at all. Presmeg's recent paper (1992) doubts the usefulness of classification into visualizers and non-visualizers. At the very least, designing pedagogies to cater to individual preferences is complicated by the difficulty of deciding which student to treat in which way and which definition of visual to use. Certainly, defining some people as "visualizers" may imply that certain individuals are closed out of understanding certain mathematical concepts which have no concrete antecedent.

The assumption is sometimes made that use of visual strategies is connected with weak mathematical ability. Eisenberg and Dreyfus (1991) have pointed out that mathematically talented students who have the ability to think visually, sometimes express "reluctance to visualize in mathematics" (p. 25). Some authors suggest that students may attempt to <u>avoid</u> visual considerations because of the role accorded visualization in our educational system (Gollwitzer, 1991; Presmeg, 1986a; Vinner, 1989). If it really is the case that weaker students are more likely to use visual methods (Battista, 1990; Lean & Clements, 1981), then this would tend to explain the observations that these methods are sometimes less effective than analytic strategies. But perhaps the most harmful, yet quite common difficulty with visualization is that students have shown a lack of ability to connect a diagram with its symbolic representation, a process some authors consider to be an essential companion to visualization (Krutetskii, 1976; Presmeg, 1986a, 1986b).

Presmeg's work suggests ways in which logical rationality is intertwined with visualization (Presmeg, 1992). Following Presmeg, we suspect that for most people both visual and analytic thinking may need to be present and integrated in order to construct rich understandings of mathematical concepts. In this regard, therefore, we prefer to discuss visual and analytic thinking in terms of strategies, approaches, and experiences rather than in terms of individual preferences or learning categories. >From a practical perspective, concentrating on specific activity rather than on competencies opens the potential for discussing this issue in a way that accounts for those situations where both visual and analytic strategies are possible, and where people seem to learn by combining the two ways of thinking.

Indeed, it may be the case that every method is such a combination and that the interesting questions concern the nature of this mixture. Our contention in this paper is that the mixture is fairly complex, and that it is important to understand how people combine visual and analytic approaches in solving mathematical problems and understanding mathematical concepts. To this end we will suggest a model that we believe illustrates possible interrelations between visualization and analysis. Following the discussion of our model, we offer some pedagogical considerations that it may suggest.

THE STUDENT INTERVIEWS

Throughout this discussion we present excerpts from interviews with students expressing their understanding of the elements and operation of D_4 . We include these data to illustrate the distinctions our definitions are intended to help us make and the cognitive development postulated in our model. As is common in this sort of research, we study the learning patterns of a number of students to postulate a learning process that may be experienced by an individual. But we are in no way suggesting that these data "prove" that our definitions and model are "correct." Rather we suggest that our definitions offer possible tools for interpreting our data, and that the data are reasonably consistent with these distinctions. We also see the student examples as useful in elaborating the interconnections between visual and analytic thinking proposed in our model. We aim for a reasonable interchange between our model (as a theoretical explanation) and our observations (as an empirical one) Each must be informed by the other, but neither is a source of complete understanding in itself. Clearly, there are other possible definitions and models that one might consider. Nevertheless, insofar as our choices make sense for these data and this problem, then it would seem reasonable to use them in studying similar problems in other contexts.

The student interviews were collected from students participating in Abstract Algebra courses at two midwestern universities. In both cases, the courses were following an experimental approach designed by Dubinsky and Leron (1994) which uses cooperative learning and computer activities as specific strategies for teaching abstract algebra concepts.[1] One of the courses was taught by Dubinsky and the other was taught by a professor who was not involved in the original course design but who had previously used a similar pedagogy for teaching calculus.

In these courses, students worked in teams to construct computer tools for examining their conjectures in specific mathematical contexts and to make inferences regarding mathematical objects. The students encountered the D_4 material through group activities using the programming language, ISETL, to construct and manipulate permutations corresponding to rotations and flips of a square. The group D_4 was discussed following other student work with permutations earlier in the course; it was presented using both analytic and visual examples.

As preparation for the interviews reported here, students had been given questions on D_4 as part of a classroom exam which included questions on other course material. On the examination, students working in groups were asked to list the elements of D_4. Although they were not asked explicitly to compute the product of elements, these calculations were required for them to perform several tasks, such as finding all subgroups. Therefore all the students had participated in groups which calculated the operation table for the elements of D_4.

Following the exam, 32 interviews with individual students were conducted by staff members involved in the courses (including the instructors). Each interview was audiotaped and subsequently transcribed and analyzed by the authors. Individual students were asked to explain the exam solutions as they looked over their own exam papers with the interviewers who prompted them where necessary to clarify their thinking about the elements and operation of D_4. Students were sometimes asked to confirm their explanations by demonstrating how they computed the product of two elements. We expected students to show us one of two ways to solve this problem—visually by manipulating a physical or mental square, or analytically by multiplying permutations.

After explaining their approach to the exam problem, students were asked to describe another way to do it, and to evaluate the two methods. Interviewers were encouraged to help students recognize the alternative method in the event that they did not see two ways of coming at the problem. In order to encourage students to talk more about their own ways of solving the problem, interviewers asked them to express a personal preference for either approach. Since the exam had been completed by a study group, we did not consider the exam strategy to represent a student's individual preference. In any case, we do not consider personal preference to be a focus of the current study.

We take these student responses to represent the ways students were dealing with the problem given their group learning environment, their own preferences, and their level of understanding of the possible procedures. Indeed, some students responded to the request to evaluate the methods with comments about their own general strategies for doing mathematics. Even though the exam papers represented group work, group members did not always describe their own solutions in similar ways in the interviews. Some students openly disagreed with their group in the interviews and described the solution in their own terms. Thus, we took each student's interview response as her or his own exam strategy. In the excerpts presented here, the names have been changed to conceal the identities of the students.

OUR WORKING DEFINITIONS OF VISUALIZATION AND ANALYSIS

This section discusses the working definitions which informed the observations discussed below. Several issues were considered in negotiating our definitions.

Does looking at a picture on paper, or a blackboard or a computer screen constitute a visualization in and of itself, or must the term visualization also involve some sort of mental constructions by the individual? Put another way, we can ask if visualization consists of the formation of images or does it consist of reasoning about images that have been formed. (See

Lean & Clements, 1981; Presmeg, 1986a, 1986b; Pylyshyn, 1973, 1979; Webb, 1979; Zimmermann & Cunningham, 1991.)

What is the opposite of visualization? Is there a dichotomy between visual and what may be called analytic thinking that can be discerned in observations, or, if the situation is more complex, then what is the relation between the two?

Is visualization concerned with the formation of mental images in reaction to the perception of objects interpreted as external, or does it consist of the construction of such perceptual objects based on mental images of the individual?

If visualization has to do with objects interpreted as external, then how do we distinguish between external or "concrete" objects and "abstract" or mental objects? For example, if we are to use the adjective "external", we surely want to apply it to a picture of a square (or a cut-out cardboard square), but perhaps not to a normal subgroup. What about integers? Is an integer an external object that can be perceived, or does it exist only as a mental construction? And is a normal subgroup an abstract object for everybody? Are there individuals for whom a specific normal subgroup is just as concrete an object as an integer—or a cardboard square? Is it possible to visualize dynamic events, or is visualization restricted to static objects?

Defining Visualization

In choosing our working definition of visualization, we are not necessarily resolving the issues raised above but rather taking a stand in relation to them. First we explain what we mean by the term and then attempt to use it only in that sense for the remainder of the paper.

Our definitions are rather narrow and there is a reason for this. It is tempting to suggest that visual thinking is reasoning that consists of mental transformations of objects which are either constructed in the mind or in some perceived external "reality." Consider, however, the following statement of Piaget: "To know an object is to act on it. To know it is to modify, to transform the object and to understand the process of this transformation and, as a consequence to understand the way the object is constructed" (Piaget, 1964, p. 176). In other contexts (Dubinsky, 1991) we have expanded this point to include the interiorizations of such transformations into mental processes and the encapsulation of such processes into mental objects. Moreover, we have conjectured that all mathematical thinking can be described in such terms. There is not much difference between this last statement and the suggestion above for a definition of visual thinking. Hence, such a choice of a definition would amount to little more than saying that visual thinking in mathematics is the same as (what we have considered to be) simply thinking in mathematics. This might not be wrong, but it would not add very much to our understanding of mathematical thought.

Our choice then is to take a fairly narrow view of visualization and then, in later sections, consider the nature of thinking that is based on visualization as we view it. In the end, we will come rather close to the position that all thinking is based on visualization and so our contribution may consist of little more than suggesting, in some detail, the nature of the role which (this narrow view of) visualization plays in mathematical thought.

We need to make one other qualification before proceeding. We will use the term "external" to mean anything which an individual accesses through sensory/motor experiences. Also, the term "event" will refer to some dynamic, external phenomenon.

Definition:

Visualization is an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses. Such a connection can be made in either of two directions. An act of visualization may consist of any mental construction of objects or processes which an individual associates with objects or events perceived by her or him as external. Alternatively, an act of visualization may consist of the construction, on some external medium such as paper, blackboard or computer screen, of objects or events which the individual identifies with object(s) or process(es) in her or his mind.

There are several remarks we would like to make in connection with this definition. We are saying that a visualization consists in the doing and not the product of what is done. Thus, a sketch of a square, or a cardboard square that has been cut out do not alone make up a visualization. We also would exclude merely looking at a picture as opposed to what the individual might make of it in her or his mind. The visualization lies in the connection that an individual makes between the picture or cut out and something in her or his mind.

When we are speaking of a construction in some external medium, a visualization does not need to be accurate. Thus, a mathematician in thinking or talking about how a subgroup is tested for normality might make helpful diagrams on a blackboard. This would be a visualization, even though the sketches would not necessarily "look like" normality. Similarly, an interpretation of the solution of two linear equations in two unknowns as the intersection of two lines in a plane is an act of visualization as soon as the simplest picture of two intersecting lines is drawn.

On the other hand, if a mathematician is imagining the construction of left and right subgroups and musing on whether they will always be equal, in the absence of any sketches or potential sketches, or if he or she were applying an algorithm to solve two simultaneous linear equations in two unknowns then, by our definition, these would not be acts of visualization.

Thus we restrict visualization to constructions that transform between mental and other media. We rule out as visualizations the construction of mental images based entirely on other mental images in the absence of external media. We would also exclude the copying of a picture from one piece of paper to another if this did not involve mental constructions associated with the paper images. It can be argued, however (Piaget, 1977; Wheatley & Brown, 1994) that such constructions are always present.

Finally, we include visualization of external events in addition to objects. We note, however, that looking at a picture and watching a race are very different experiences, and the mental constructions made in reaction to dynamic phenomena are much more complex than those made when perceiving a static object. Nevertheless, we consider the mental constructions that are required in order to "perceive" the rotation of a square to be just as much a visualization as to perceive a square at rest through visual stimuli.

Defining Analysis

It would be simple to say that analytic thinking is anything other than visualization. But such a definition would not be very helpful since the total universe over which one is taking the complement is not necessarily well understood. We prefer to say explicitly what we consider to be analytic thinking and leave open for now the question of whether there is thinking that is neither visualization nor analysis.

We want to avoid confusion of our use of the term analysis with its more common meaning, as when a chemist might analyze or break down a compound to discern its component parts. Our use of analysis may be better illustrated by the biologist who analyzes the nature of a plant through decomposing it into its parts, as well as thinking about the relationships among those parts and synthesizing them into various other wholes such as leaves, flowers and seeds. Thus, we include naming of parts in our view of analysis, but we also include intellectualizing about problems through thinking about relationships among those parts and synthesizing them into various new wholes.

Definition:

An act of analysis or analytic thinking (we will use the two terms interchangeably) is any mental manipulation of objects or processes with or without the aid of symbols.

In connection with this definition, two remarks come to mind. First, we are including here the mind-to-mind constructions that were omitted in our definition of visualization. These constructions include the reflective abstractions of interiorization, coordination, reversal, and encapsulation which, one of the authors has conjectured, might be capable of describing all mind-to-mind constructions (Dubinsky, 1991). In particular, we include what is usually referred to as logical analysis and reasoning.

Second, the use of symbols in an act of thought presents a very fine line between analysis and visualization. When the symbols are taken to be markers for mental objects and manipulated entirely in terms of their meaning or according to syntax rules, then we take the act to be one of analysis. When, however, the nature of the symbols themselves and/or their configurations is used then we would consider it to be an act of visualization. Obviously, there are situations in which symbols are used both analytically and visually, in possibly complex combinations. Later we will see that combinations of visual and analytic acts are very common.

Examples of Student Approaches to the D_4 Problem

In this section, we present excerpts from the transcripts of interviews illustrating our definitions of visual and analytic methods. In addition to identifying various comments as indicating visual or analytic thinking according to our definition, we consider how separate are the occurrences of these two strategies.

When asked to explain her thinking about the exam question, Vicki indicates her visual strategy in that thinking about transforming a square is, for her, more reliable when expressed as physical motions of a physical square.

Vicki: We're wonderful with this, we'd make a little square, R_0 is we'd number all of our corners and we'd say "look we're going to do R_{90} and then H, we'll make a 90 degree turn and then a horizontal flip and see where we end up." That would be, that was the easiest way to see exactly what it was going to be without making mistakes because normally if I tried to do it without that I made a mistake. It was harder to see that way.

According to our definitions, Vicki is describing visual thinking when talking about the physical square, and analytic thinking when referring to the labeling which is a manipulation of this physical object.

In responding to the same question, Victor performed transformations on a mental image of a square. He described a strategy in which he used color to keep track of the corners of mental images of squares as he transformed them in his head.

I: When you work with D_4, how do you go about calculating the product of two elements?

Victor: I would just do it in my head. [...] I picture the square. I, like, in my mind I mark one specific corner with color, in the square, in my mind. And this like one color on one side and that same corner is a different color on the other side of the square or whatever, so I can keep track of which side is facing me, what rotation it is. It's easy.

Victor appears to be describing an act of visualization in the sense that he has constructed, in his mind, a process which he seems to associate with the motion of a physical square. On the other hand, this example of visual thinking also has an analytic aspect in that he colors one corner (in his mind) "so I can keep track.. .." This suggests that color, which is normally considered to be visual, is being used here by Victor as a symbol to help him manipulate his mental image of a square.

In the following excerpt, Amy does not mention the physical square, but emphasizes the cycle notation in her version of how she works with the problem.

I: How did you actually do the computation of the products?
Amy: Well, I always have to write everything down. So I went ahead and wrote out the cycles, and I wrote out the permutations and went through and multiplied them out—or whatever you call it. I don't know what you call it. I did the operation on it.

Amy's use of permutations (in cyclic notation) appears to be entirely analytic, but this is not always the case. Anthony, a groupmate of Victor describes a method that is mathematically identical to that of Amy, but also involves a measure of visualization.

I: How did you go about calculating the product of two elements in that group?
Anthony: [...] taking the two elements in cyclic notation, you operate the one on the left to the one on the right, and you find the one and see what it corresponds to, maps to.

Notice that in describing his analytic manipulations of permutations, Anthony makes use of spatial terms such as left, right, and maps to. We see this as an example of the visual aspect of using symbols.

In the following excerpt, Harold's use of visual and analytic thinking appears more balanced than we have seen in the previous examples.

I: Tell me how you went about computing these products.
Harold: Well, uh, this problem involves the symmetries of a square, so we just made a square out of paper, and uh, what we, we performed the permutations of that square in order.
I: All right, here's a square. Why don't you show me just what you did.
Harold: Do you have a pen or something? OK. Um, let's see, I'm just gonna label these corners going clockwise, 1 through 4, and write it on both sides. I have to be careful that I get the numbers to match up. And I'll mark this as the home position, uh R_0. So if I want to compute say, R_2H, I first perform, uh, two clockwise rotations and then a horizontal flip and uh, I look at this and I see what this would be equivalent to—I've got 1234, uh, this would be equivalent to a vertical flip so, uh, I'm going to try to fill in under R_2 and H, a V for vertical flip.

Harold's actions with the square show a conscious analytic strategy for making sure the positions of the numbers are accurate, while his language refers consistently to both kinds of representations.

These examples show that our definitions can be used in a fairly unambiguous manner to distinguish between specific examples of what we are calling visual and analytic thinking. Performance cues for a visual categorization include use of a physical square, reference to a mental one, reference to rotations and flips of the square. Evidence for analytic strategies might include working with permutations, cycles, and symbolic manipulation. Sometimes students also told us about their own general strategies for doing mathematics by making statements such as, "I have to see things," or "I find it easier working with numbers than with objects."

However, the presence of both analytic and visual components in almost every students' response is further support for the position, referred to earlier, that although specific instances of thinking can be categorized, it may not make much sense to attempt to classify an individual as either a "visualizer" or an analyzer."

A "visual solution" in our situation generally involves the analytic act of labeling the square and the analytic determination, from the final position of the square (as a result of two transformations), of a single transformation. The "analytic" solution also involves interpretation of elements of D_4 as permutations, a process which requires at least some connection to the square as a visual element. We agree here with Piaget (1966, 1977) and Presmeg (1986a, 1986b) that there seems to be little visualization that does not contain some analysis, and we extend that argument in this case to its reverse—there seems to be little analysis without some use of visualization.

Even though both strategies may be available to a student, he or she can often have difficulty make connections between them. Our observations reveal that mixing, harmonizing, synthesizing the strategies usually show students' more mature understanding of the problem. This suggests that there may be a more important question than that of classifying an individual or even a strategy. It may be more fruitful to try to understand the interrelationship between visual and analytic approaches that a student may have constructed in her or his mind. To this end, we consider in the next section a model of how visual and analytic thinking might interact as an individual grows in her or his understanding of a particular mathematical situation.

A PERSPECTIVE ON VISUALIZATION AND ANALYSIS

In this section we describe a model for thinking about mathematics that is a synthesis of visualization and analysis. We will consider this model in light of some of the issues raised in the previous sections.

Rationale for a Model that Synthesizes Visualization and Analysis

In constructing a model of how visualization and analytic thinking can be combined, we are influenced by various themes in the literature as well as by the data we have gathered in observing individuals trying to solve the above D_4 problem. Our model has strong roots in the work of Piaget on perception (Piaget, 1969). Specifically, we consider our model to be a natural extension of his analysis of the interdependence of perception and intelligence.

It is important to emphasize that our model is not intended to be a definitive description of how people think about problems such as the ones we are considering in this article. In particular, we are not suggesting that our data in any way show that this model is "correct," nor are we proposing that students should be forced to think in this way. What we are suggesting is that this model might be an approximate description of how some people learn about these kinds of problems. When we see an example of someone not adhering to this model, then it is either a case of the individual having difficulty in thinking this way and possibly requiring assistance, or a case of the individual thinking in some other way. If the former occurs often enough, our model should be useful in suggesting pedagogical strategies for the mathematical topics we are considering. To the extent that the later occurs, our model may require further thought and revision.

Thus, our criteria in this paper for deciding on the value of the model are whether it is a useful tool in thinking about the issues raised in the literature and whether it is useful in organizing and making sense of the data we have gathered. It is part of our overall research agenda to evaluate our models through instructional treatments whose results enable us to further refine those models, or even eventually to abandon them.

Piaget's rather extensive empirical experiments and his analyses of them provide strong support for the more recent conclusions of Krutetskii (1976) and Presmeg (1992) that all visual thinking includes a logical aspect. In fact, this goes back to one of the very earliest aspects of intellectual development—simple perception. According to Piaget, even the perception of static objects involves dynamic action on the part of the viewer. He or she receives a large number of visual stimuli by moving eye-attention to different parts of the object, usually according to patterns such as looking more often at the center, at special features, etc. and organizes these stimuli to form the perception. Thus, even at this primitive level, it is the intelligence used in organizing the stimuli that determines what is perceived. Moreover, we see that even though an individual may be viewing a static object, the act of perception itself involves motion on the part of the viewer. (See Piaget, 1969.) This dynamism of imagery has been remarked upon by several authors (e.g., Arnheim, 1969; Hebb, 1968; Neisser, 1967).

Dynamic events present a more difficult problem than do static objects. Indeed, Piaget concludes from his experiments, that one cannot directly perceive a dynamic event. What happens instead is that one takes discrete, mental snapshots and "achieves the illusion of continuity by a diffuse consciousness of the jumps leading from one snapshot to the next" (Piaget, 1977, p. 677). Thus, according to this view, it might be said that one does not actually see motion but rather one "makes it up" by applying intelligence to a set of static visual stimuli. Again, there is support in the literature for this view (Hebb, 1968; Neisser, 1967). Of course, one might argue that in viewing <u>either</u> static objects or their motion, the individual applies intelligence to coordinate a number of different snapshots taken from different positions of the eyes and in the case of motion of the object(s) in their successive positions. Nevertheless the perception of motion is still more complex than that of a static object in this formulation because there are two levels of transformations to coordinate.

Piaget extends his analysis to the use of mental images. His experiments suggest that mental images form the raw material out of which operations (essentially the same as mental processes in our context) are built. He goes on to articulate the issue that forms the main starting point for our model:

> "The conclusions to be drawn from this type of experiment, as from the [seriation of lengths] are therefore first that operations transcend images, while making use of them;

> and second that, once brought into play, operations direct images and even determine them almost entirely in some cases". (Piaget, 1977, p. 681).

In a sense, our model does little more than extend this last analysis to higher levels of thinking and stretches it from two steps to a multi-step procedure.

The VA Model

In describing our model, which we will call the Visualization/Analysis or VA model, we make use of Figure 2. We express this model in terms of discrete levels of visual and analytic thinking. This is, however, at best an approximation of what is very likely a continuum. We feel that certain points in this continuum are of particular importance and it is useful to focus on them in order to describe what we take to be a generalized development pattern. Discrete approximations of the continuous are fairly common and very useful in many branches of mathematics and science. A physicist, for instance, may describe electrons as orbiting the nucleus of an atom at discrete distances despite understanding that they may actually appear at all positions, although not uniformly. Such approximations enhance discussions of nuclear structure in much the same way discrete terms help us to discuss our model. The thinking strategies specified by our model should be thought of as continuous. Indeed, these strategies, when observed, may even appear to be occurring simultaneously.

The thinking described by our model in Figure 2 begins with an act of visualization, V_1. This could consist of looking at a computer screen, or a drawing or some other "picture" and constructing mental processes or objects. Or it could consist of making a picture or some other external means of expressing a collection of mental objects and processes.

{Insert figure 2 here}

The next step is an act of analysis, A_1, which consists of some kind of coordination (in Piaget's sense of reflective abstraction) of the objects and processes constructed in step V_1. This analysis can lead to new constructions.

In a subsequent act of visualization, V_2, the student returns to the same "picture" used in V_1, but as a result of the analysis in A_1, it has changed. That is, V_2 could amount to an application of A_1 to reinterpret the picture or it could lead to the construction of a new picture to replace the old one. In any case, the result is an external representation in which the individual achieves some richer understanding of the original situation. As the horizontal motion in the model is repeated, each act of analysis, based on the result of the previous act of visualization is used to produce a new, richer visualization which is then subjected to a more sophisticated analysis.

Eventually, as indicated in Figure 2, the acts of visualization and analysis become, for the individual (but not necessarily for the observer) successively closer. This is meant in at least two senses. At first, acts of visualization and analysis may be seen by the individual as being very separate and quite different. The passage from one to the other may represent a major mental effort, which can even be resisted in some cases. Gradually, the two kinds of thought become more interrelated and the movement between them becomes less of a concern. Part of the reason for this development is the second sense in which the two kinds of acts approach each other. In the beginning, the individual may interpret visualization as relating to events that are mainly external to her or himself (as in picturing a physical square, or drawing a picture on the blackboard), whereas analysis may be seen more as originating from the individual (as in inventing and using a symbol system). As the acts continue, both kinds of thinking become more and more deeply interiorized within the mind of the individual. After some time, closer to the top of the VA triangle, analysis and visual understandings are synthesized so that it may be very hard for the individual or the observer to distinguish between them.

We emphasize that this varying "distance" between visual and analytic thinking, embodied in our use of a cone, refers not to the viewpoint of the individual, but to how we as investigators might interpret the situation. For example, as we saw above, Victor expresses his use of color in terms of how he can "picture the square," a visualization. In contrast, we see his use of color as a kind of labeling, which we would interpret as an analytic strategy.

All activity continues until the individual feels that he or she has re-equilibrated from the disequilibration that caused her or him to begin thinking about the situation. This could happen because a stated problem was (thought to be) solved or a feeling of confusion was replaced by an impression that greater enlightenment had been achieved. Of course, the individual can also choose to abandon the effort altogether.

An Example.

Let's return to our D_4 interviews discussed earlier, but this time from the point of view of our VA model. We will consider how the students' thinking about the two problems in this situation can be described in terms of explicit V_i's and A_i's as specified by the VA model. We emphasize that this is one way of describing the students' thinking. We are not suggesting that students consciously think about such matters, or even that our description is, in any sense, "correct." We only suggest that it does fit reasonably smoothly with the data and hence may provide a useful language for discussing student thinking. Moreover, we expect that this model may offer a framework for thinking about pedagogical strategies to help students develop the ability to solve problems such as these. We begin with the problem of determining the elements of D_4.

The visualizations at V_1 might consist of cutting out a paper square and performing various physical actions of moving the square around without much in the way of restrictions.

> Kendra: Since, D_4 is a repositioning of a square, we took a sheet of paper, labeled the corners and just moved it around.

In the following excerpt, the student mentions flips and rotations, but seems to be willing to accept any motion as long as the square is "still a square."

> Faye: Well, uh, it's like—oh gosh, I don't know how to say it—um, well it's the operations on a square like rotations and I don't know. [laughs] So that whenever you do the operation it will still be a square. If you flip it any way, it's still a square. If you rotate it, it's still a square.

We consider these reactions to be visualizations in the sense that the idea of D_4 and its elements is expressed entirely in terms of the physical square and unstructured motions of it.

The act of analysis A_1 might arise out of thinking about restrictions on the "allowable" motions of this square. For example, the student might introduce the restriction that a motion of the square begins and ends with the square in the same position.

> Andrea: OK. Um, reading ... First of all I always, like, have to look at it and go, "D_4, that's when you take the square and flip it around. And, that's all the positions that it can land up in so that it's still, you know, corners in the same places and all that."

The mental manipulations in this case consist of imagining possible motions of the square visualized in V_1 and applying the action of deciding whether it returns the square to its original position. The manipulations appear to take place without using symbols. The student may just think about a fixed position that is a frame for a square together with an actual square and imagine putting the square into the frame after moving it around. The restriction is that the square must fit into the position and the student begins to think about the movements that have this property.

The act of visualization V_2 might consist of physically performing various motions of the square that meet the restrictions determined at A_1. Usually, labels are mentioned but they are not used beyond helping to keep straight the initial and final position. In the following excerpt, the visualization, in our sense, is explicit as the student identifies acts of moving the piece of paper with "doing it in our head".

Alisa	And we actually had a piece of paper with a, b, c, and d, and we would do it. We had to rotate it once, rotate it, and then flip it.
I:	OK.
Alisa:	We would do exactly what it said and
I:	You did it geometrically. OK, can you think of another way to do it?
Alisa:	In your head. We tried just like doing it in our head at first.
I:	In what sense in your head?
Alisa:	Just like visualizing it in your head, like OK, turn it and then you flip it.
I:	OK, but what did you do first of all?
Alisa:	We labeled the corners a, b, c, and d and like this one, like a, b, c, and d were like its original position with R_0 and if you rotate it once, that's what it was, then you would have your corners and draw what it would be and if you would flip it once where your corners would be in the original position.

This is an act of visualization that is similar to V_1 in its use of the piece of paper. But the activity goes beyond the relatively random motions of the early case in its requirement that the corners are in the original position. Here we see how the analytic act in A_1 affects and enriches the visualization.

The analysis at A_2 arises out of a need to organize the allowable motions which were performed at V_2 . The individual might bring to bear some analytic structure such as labels, axes, or degrees of rotation,

Ken:	By labeling the corners of a square 1,2,3,4 respectively, starting at the upper right hand corner, and we knew that, uh, the repositionings were your four rotations — each 90 degrees greater than the previous one.

I: In which direction?
Ken: Clockwise is how we took it, I think that was defined in the book. And then we knew that you had the horizontal, vertical, and the two diagonal flips that you had to consider. And so we, you know, had labeled a square. On the reverse side of the square we had labeled it with the same numbers, uh, so that, you know, the same corner had the same number on it.

A visualization V_3 could then consist of performing motions using some analytic structure to keep things straight. The following student, for example, mentions the difficulty of imagining the motions without some structure. Note that she explicitly credits the labeling and thinking about permutations for making possible the visualizations she does make. In this sense, the visualization at V_3 is based on the analysis at A_2.

Karen: Oh, what we did, we drew out a square and we labeled it and we did it visually so, 'cause we couldn't imagine it in our head. It's hard for us to think like that. So, what we do is we cut out a square and rotated as... Then we say, "OK, this is the same thing as this." And we wrote all of the R_0, what is was, down — the numbers and what permutations form of how we label each side. And then, whenever we got back to that permutation again, we said, "OK, that's the same as R_{90}", if it was, or something like. That's how we did that.

The visualization here is more sophisticated than it was in V_1 and V_2 in that the individual is guided by the analytic structure introduced in A_2 when manipulating the square.

A_3 could then consist of realizing that there are precisely eight ways of performing a motion that meets the restrictions. These might be named in some way. Again this analysis relies on the visualizations and analyses that have preceded it.

Raj: You're talking about symmetries here, when you're discussing this particular set. And the way you can obtain the different symmetries is you look at a square. And you try to see all the different axes by which that square can be replaced such that it will give you a different ending position, that compares to the initial position. And so, you started, you took an operative square, labeled the corners 1,2,3, and 4 just so that we could keep track of the corners so we know when we came upon a different position. And then we do the whole thing for rotations, and the same way—vertical. I mean, suppose you had a rotation of 90°. Those were these 4 and then there would be additional 2 more rotations, given by the horizontal and vertical and finally was the diagonal, the 2 possible diagonals.

And,

Nicholas: You've got four different rotations, four different flips. Basically, they're on the horizontal and vertical axis, uh, two diagonals, and rotating, and, uh, they're either 0, 90, 180, or 270 degrees...you get eight elements in D_4.

The reader will notice that in the last few excerpts, it is not always easy to distinguish between visual and analytic thinking. This is partly because the individual is not only thinking about the present activity, but also going back and forth between it and the previous analyses and visualizations on which it is based. We feel there is another reason for this. As understanding of a situation becomes more sophisticated, the visual and analytic thinking which rely on each other also become more highly integrated. This is why we have a triangle in Figure 2 rather than a rectangle or cylinder.

At this point, the first problem is solved and the elements of D_4 have been listed. The naming of the symmetries can occur because (or is the cause) of the fact that the individual has encapsulated these motions to be objects. Most of the students did this but, of course, since D_4 was considered in the course and standard names were used very often, it is impossible to see, in these examples, how the students go about constructing in their mind the connection between the symmetries and the names.

We can, however, look at how some students dealt with a further step—the association of a permutation with each symmetry. This occurred in the context of the second problem in this situation: working out the product of two elements of D_4. To this point in our example, it is reasonable to expect that, for many students, the development followed something like the sequence V_1, A_1, V_2, A_2, V_3, A_3. Note here that the discussion of permutations in the interviews happened at different times for different students, sometimes as the result of a prompt from the interviewer to "think of another way to do it." Also, in the interviews, some students made a clear shift from describing visual strategies to discussing analytic thinking, while others discussed the two in a much more interconnected way. Consider, for example the following excerpt.

Tony: So this is where you started with and it ends up, let' see (pause) um this here is just a, OK this here has just been turned 180 degrees, so then what I do if I was taking this one with this one, is just turn this one 180.

I: OK.

Tony: So then it would become, the 4 would end up there in the bottom position. The one next to it, gets to go up there and the 3 over there.

I: OK, so you are working basically with just turning the square geometrically.

Tony: Uh huh. Sure.

I: Can you think of another way to do this?

Tony: Um, if you organize it in terms of permutations, then you'd write this here as the permutation, you'd use this as the identity. 1 here has gone to 4, 2 has gone to 1, 3 has gone to 2, and 4 has gone to 3.

Tony first visualizes the motion of the square and then begins to think analytically by naming the vertices and moving the numbers. His reaction to the prompt is really a continuation of this thinking in which, instead of numbers going to positions, he speaks of numbers going to numbers, which is the beginning of the idea of permutation. For convenience in discussions, we may call this sequence $\underline{V_4, A_4}$, although these indices may not indicate the order for every student.

Some students express these two steps all mixed together as one. Karen's discussion cited before illustrates how analytic and visual thinking can become intertwined as the mathematical issue becomes more sophisticated.

> Karen: So what we do is cut out a square and rotate it as... Then we say "OK, this is the same thing as this." And we wrote down all of the R_0, what it was, down — the numbers and what permutations — from how we label each side. And then, whenever we got back to that permutation again we said "OK, that's the same as R_{90}", if it was, or something like that.

We will continue our numbering system as V_5,A_5, by considering how the students deal with composing two symmetries to form a new symmetry. However, we use these numbers more for convenience of discussion than for any sequence we claim to have observed.

$\underline{V_5}$ consists of trying (often without success—it is not so easy) to perform two successive symmetries and identify the resulting symmetry by manipulating a square. In the following excerpt we see the student doing just this. The permutations are present, but they appear to be used only for identifying the results of visualizing a symmetry.

> Ron: Um, OK well you start at R_0, so you get 1, and if the two elements were 180 and H, R_{180} and H, you start by doing R_{180} so this would go to (3,4,2,1) and then you would do H so it would become (2,3,1,4).

Some students have great difficulty if they rely on visual means to compose symmetries and, like Tim, may not even get an answer they feel comfortable with.

> I: Let's do the D' and follow it with the R_{270}.
>
> Tim: OK, we have the D'.
> I: Uh-huh.
> Tim: And, u, so we, I basically in my head do this box where we have (1,4,3,2) and, uh, then (1,2,3,4) and then go back to R_{270} and 1 corresponds to 1, we have 1...
> I: Uh-huh.
> Tim: Uh, 2 corresponds to 4, 3 corresponds to 3 and 4 corresponds to 2. All right...um...hold on...here its, I'm sorry...so...

In A_5 the individual uses the permutations as an analytic tool for thinking and talking about the symmetries.

Connie:	I could write down what R_{90} was in permutation form.
I:	All right, why don't you write that down.
Connie:	R_{90} was (4,1,2,3). And then V the vertical flip was (2,1,4,3). And then these are...
I:	Why don't you compute that for me to see what you get?
Connie	Permutation, um 1 is mapped to 2, 2 is mapped to 1. 2 is mapped to 1, 1 is mapped to 4. 3 is mapped to 4, 4 is mapped to 3. 4 is mapped to 3, 3 is mapped to 2. So I get (1,4,3,2), that turned out to be D.

After using various methods to compute the product of two elements of D_4, some students attempted to solve the problem both ways and compare their answers. In many cases they got different answers which were inverses of each other. The error they were making turned out to be rather subtle and is worth a separate analysis (Zazkis & Dubinsky, in press).

Remarks about the VA Model

In introducing the VA model we include, even at the lowest level, the possibility of visualizing dynamic events. As we indicated earlier, this cannot be done directly through perception. It must be the result of a combination of visual and analytic thinking. Whether the event involves motion of physical objects, or transformations of mental objects, construction of a visualization of it can be described by the VA model.

The VA model offers a simple explanation for the apparent relation, reported in the literature, between people who appear to prefer visualization and those whose mathematical abilities are weaker (Presmeg, 1986a). If our model is right that thinking about this problem begins with visualization, then the weaker the student is, the less likely he or she is to progress very far. Obviously, experiments would be necessary to establish it, but this point could account for at least some of the apparent preference for visualization by the weaker students.

In fact, the VA model calls into question the whole notion of a visualization/analysis dichotomy. We have already seen that some authors do not accept this in its simplest form and consider that all visualization involves analysis, or that there is a continuum between visualization and analytic thinking (Krutetskii, 1976; Presmeg, 1986a, 1986b; Suwarsono, 1982).

Using the VA model, we can suggest yet another possibility. It could be that a preference for and difficulties with visualization (Bishop, 1986; Tall, 1991; Vinner, 1989; Goldenberg, 1991) is no more than an individual's tendency to dwell on one side or another of the triangle, for example, when communicating her or his thinking. An individual might be more comfortable drawing pictures, or writing formulas, but that does not change the fact that he or she needs analytic thinking in determining what to draw, or eventually constructs a rich mental picture that determines what symbols to write.

If the VA model is an approximation to the way some people think about some problems, then it is hard to see how one could use those problems to see if those people were "visualizers" or "analyzers." Consider the example of thinking about D_4 described here. Does it really make sense to classify as a visualizer, someone who is working at V_1 and does not do much more than draw a square together in the same category with someone who is working at V_5 with a labeled square and is computing the composition of two rigid motions? Similarly, when we speak of someone who prefers analytic strategies, are we satisfied that this groups the individual at A_1 who is trying to figure out what motions are permitted with the person at A_5 who is using (and understanding the geometric interpretation of) procedures for multiplying two permutations? Indeed, would we want to consider the processes of the latter individual as similar in any way to someone who uses the same formulas to make the computations just as quickly and accurately, but has no idea of their meaning in terms of rigid motions and symmetry? Our view in proposing the VA model is that, although visual and analytic thinking are very different and can be distinguished by an observer, the two modes of thought are intermingled in complex ways. It is this entanglement that makes us feel not very surprised that many authors report difficulty in deciding who is a visualizer and who is not. (See Hadamard, 1925; Krutetskii, 1976; Menchinskaya, 1969; Poincaré, 1963; Richardson, 1977; Walter, 1963.) Our goal for the VA model is that it will help, not to classify individuals, but to understand the relationships between the two modes of thinking.

Finally, we would comment that difficulty in moving from visualization to analysis at the lower portions of the triangle in Figure 1 form another explanation for the resistance to visualization reported by some authors (Ferrini-Mundy, 1987; Monk, 1988; Swan, 1988; Vinner, 1989) and which we also observed (although not in any of the excerpts included here).

In a previous section we illustrated students' approaches to the D_4 problem and found that most of them had both visual and analytic arguments. Revisiting these examples through the lens of the VA model, we can see that Victor and Vicki and Harold enrich their visualization with the analytic structure of marking the corners of the square with either color or a number label. On the other hand, Amy's analytic approach is based on a previous visualization of generating the elements of D_4 and Anthony is helped in his analytic method of manipulating by visualizing the relative positions of the individual symbols.

We suggest that it might be useful to focus instruction on the different ways in which students coordinate (or don't coordinate) visual and analytic modes of thought. This leads into the final considerations of this paper in which we offer some first thoughts about pedagogy that seem to be suggested by the VA model and how it could help us look at student learning.

PEDAGOGICAL CONSIDERATIONS

The analyses in this article suggest to us the consideration of pedagogical strategies for dealing with visual and analytic thinking that are somewhat different from those usually employed. We would like to suggest such an alternative approach, but make no claim for our proposals other than that they seem plausible in light of the analyses in this paper. Of course, it would be necessary to implement them to see if they appear to make any difference.

In the VA model we have considered visualization and analysis as two interacting and mutually supporting modes of thinking, rather than as two sides of a coin, or as a dichotomy or continuum. The traditional perspective emphasizes the differences between the two modes and focuses on visualization and analysis as individual characteristics or as a "personal learning style". In most cases (since it is often assumed that most people's learning styles are more visual than analytic) this means providing visual tools for "visualizers" (e.g., Eisenberg & Dreyfus, 1991).

Presumably it also entails providing analytic tools for "analyzers." Following the VA model we propose that pedagogical approaches be shifted from emphasizing the differences between the two modes of thought to focusing on the interactions between visual and analytic thinking. It is consistent with Presmeg in a belief that "as many domains of experience as are relevant should be implicated in the mathematical learning process" (Presmeg, 1992, p. 607).

The gist of our proposal is to develop individual pedagogical strategies for particular educational situations that reflect something like the triangle in Figure 2. That is, although the ultimate goal in understanding a concept is to move vertically from the base towards the vertex, the best way to do that might be to travel horizontally with a small vertical component. This means that if an individual's thinking about a problem situation is, at a certain point in time, describable in terms of one of the analytic approaches, say A_i, then the best way for that individual to proceed, might be to move across the triangle towards V_{i+1}. Thus, pedagogy might be designed to encourage this individual to visualize, in some manner or other, the analytic activities at A_i. Similarly, if an individual is thinking at, say, V_i, then pedagogy might aim at getting her or him to analyze what is being visualized.

There are two things we are calling for in such an approach. One is to emulate the skier climbing a hill and move up by moving across. The other is to make explicit the fact that the analysis at A_i relies on the visualization at V_i and what is visualized at V_{i+1} is enriched by the analysis at A_i.

We do not deny that this approach could entail asking students to adopt strategies at variance with what they are most comfortable with. Learning is often like this. The important point is that the goal should be for students to develop a rich synthesis of visual and analytic thinking. Several pedagogies have been specially adapted to encourage visual learning as an aid to more traditional analytic approaches. Eisenberg and Dreyfus (1986, 1991) argue in several papers that reasoning about visual models of mathematical situations can greatly enhance problem solving. And a variety of studies (see Bishop, 1989, p.13 for a partial list) present evidence of the facilitative and beneficial effects of using computer generated visual images in developing understanding of mathematical ideas.

There may be students on either side of the triangle whose mathematical performance is not very strong. But weak students who prefer analytic strategies may be more successful in school than weak students who prefer visual strategies since many school practices privilege successful imitation of algorithms. However, that success may evaporate in college work where more profound understanding of material is required. Again, we suggest that "moving across" in order to "move up," at a rate appropriate for those students, may help them to make the connections necessary for proceeding to the next step.

We do not mean to imply that moving across the model is either easy or convenient for all students, especially when "analyzers" are asked to draw pictures and explain them or "visualizers" are asked to capture their images with formulas and manipulate the symbols according to logical rules. But we do think it might facilitate a richer and more useful understanding of complex ideas as a result of the effort. Many students resist working with strategies they find uncomfortable and/or challenging. We suspect this motive may be involved in the choices students like Anthony were making when they tell us as he did, "[the visual strategy] would make me understand what I'm doing more. But it's slower and more tedious."

In summary, we are suggesting that pedagogy must rely on investigations that lead to fine grained analyses of students dealing with problem situations to suggest specific points at which individuals who are thinking visually (respectively analytically) might be helped to enhance their understanding and move forward in solving problems by employing appropriate analytic (respectively visual) strategies.

Although we believe that thinking in terms of the VA model will turn out to be helpful in this pedagogical endeavor, we have really not said anything at all in this paper about just how one might go about developing specific instructional treatments along the lines of our suggestions. This will require major curriculum development efforts. But perhaps we have provided some food for thought for those who will make such efforts.

SUMMARY AND CONCLUSION

Our study suggests a perspective on the issues of analysis and visualization, presenting their interrelationship as a symbiosis rather than a rivalry of opposite poles. Our model suggests a further refinement of Krutetskii's (1976) and Presmeg's (1986) writings, where "visual" and "analytic" axes are orthogonal. Their proposal of orthogonal axes presumes a unique intersection point between these two ways of thinking, while the VA model suggests multiple intersections.

We used the problems of finding the elements and their products in the dihedral group of order 4, as a tool in our investigation. While some approaches to the problem appeared to be primarily visual or primarily analytic, we found analytic components in visual approaches and vice versa. In the VA Model we attempted to show how visual approaches benefit from analytical thinking and how analytical approaches are enriched by visualization.

We are encouraged by what appears to be a good fit between our formulation and the data. Future research is necessary to study the suitability of the VA model in other problem situations and to other mathematical topics to examine the extent of its descriptive and explanatory power for examining students' learning.

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Footnotes

[1] The pedagogy used for our observation sites is described in an article by Leron & Dubinsky to appear in the 1994 American Mathematical Monthly called "An Abstract Algebra Story."

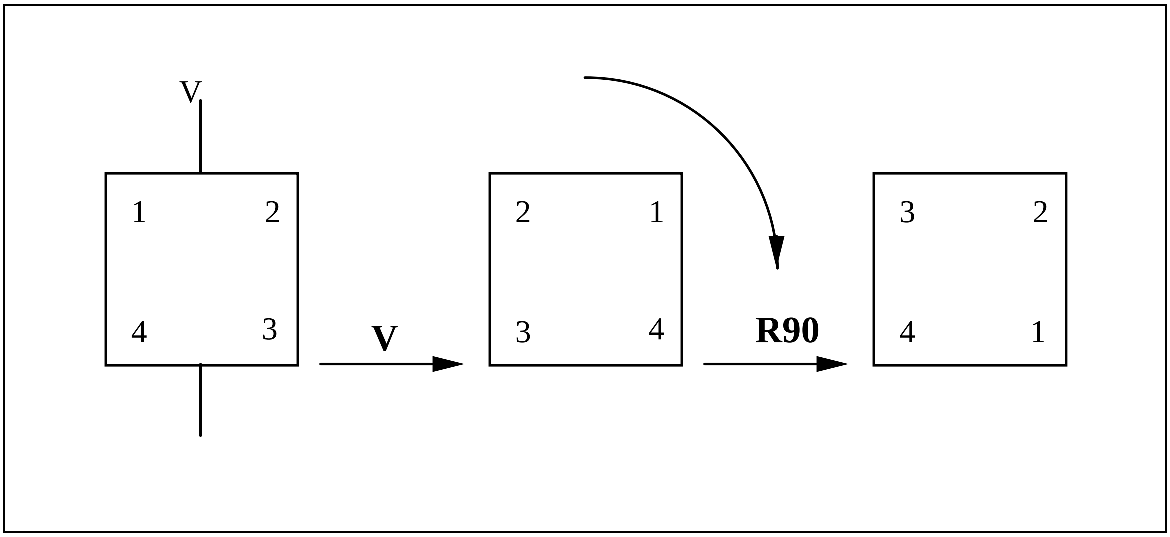

Figure 1. Vertical flip followed by a 90-degree right rotation.

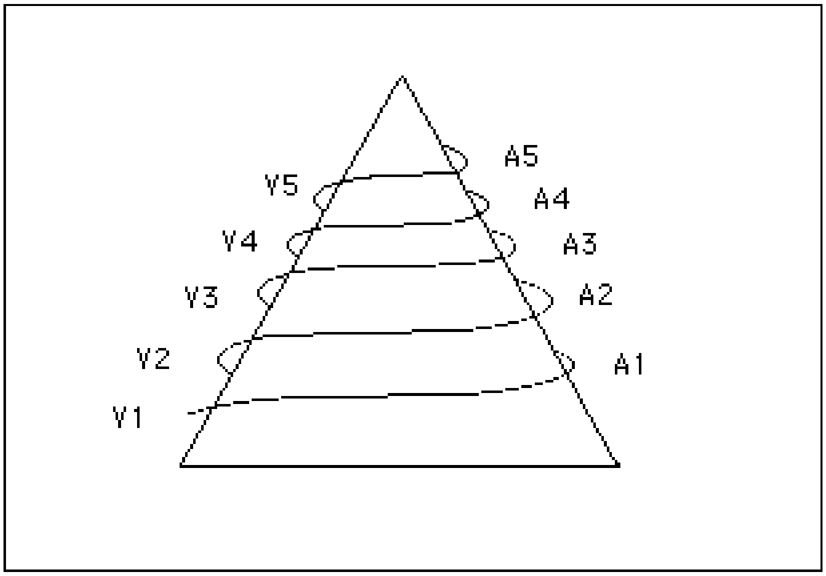

Figure 2. Visualization Analysis Model

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