Summability in Topological Spaces

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Outline

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- Fejer's theorem on convergence of Fourier series.
- Komlos' theorem for L^1-bounded sequences

And so on $\cdots$.

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More precisely, consider the following classical summability methods.

- (a) Strong convergence,
- (b) Statistical convergence,
- (c) Distributional convergence,
- (d) classical matrix summability.

A-Strong and A-Stat Convergence

Thoughout assume that $A = [a_{nk}]$ is a nonnegative regular summability method. Not much loss takes place to assume that the row sums equal to one.

Definition (A-strong convergence)

We say that $x = (x_k)$ is *A-strongly summable* to α if

$$\lim_{n\to\infty} \sum_k |x_k - \alpha|\, a_{nk} = 0.$$

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Definition (A-stat convergence)

We say $x = (x_k)$ is A-statistically convergent to α if for any $\epsilon > 0$, we have

$$\lim_{n\to\infty} \sum_{k:|x_k-\alpha|\geq\epsilon} a_{nk} = 0.$$

A-Dist Convergence & A-Suammbility

Definition (A-distributional convergence)

If x is a real sequence, we say x is A-distributionaly convergent to F, where F is a probability distribution on $\Re$ and

$$\lim_{n \to \infty} \sum_{k : x_k \leq t} a_{nk} = F(t),$$

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Definition (A-summability)

Finally, we say that x is A summable to α if

$$\lim_{n\to\infty} \sum_{k} x_k \, a_{nk} = \alpha.$$

A-statistical convergence

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The remaining two, A-strong convergence and A-stat convergence, use distance structure since they both use

$$\|x_k - \alpha\|, \qquad \rho(x_k, \alpha).$$

This then leads one to consider general topological structures by replacing $\rho(x_k, \alpha) \geq \epsilon$ by its natural counterpart,

$$x_k \notin U_\alpha,$$

where U_α is any open set containing α. So, how do you bring the summability

structure into the topological space?

Mathematical Structure

Let $(X, \mathcal{B}, \tau)$ be any topological space, where $\mathcal{B}$ is the Borel sigma field generated by the open sets. In order to define a summability notion in X, we will inject several probability measures μ_n defined over $\mathcal{B}$ with the help of a nonnegative regular summability matrix $A = (a_{nk})$.

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Consider $([0,1], \mathcal{M}, \lambda)$ be the usual Lebesgue measure. Partition the interval $[0,1]$ by $A_{n,0} = [0, a_{n0})$, and

$$A_{n,k} = \left[\sum_{j=0}^{k-1} a_{nj}, \sum_{j=0}^{k} a_{nj}\right), \qquad k = 1, 2, \cdots .$$

Let $f_n : [0,1] \to \mathbb{N} := \{0, 1, 2 \cdots\}$, where $f_n(\omega) = k$ for $\omega \in A_{n,k}$. Over the sigma field of powerset of $\mathbb{N}$ this f_n induced a measure ν_n defined by $\nu_n(k) = \lambda(A_{n,k}) = a_{nk}$.

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Any function $x : \mathbb{N} \to X$ is automatically $2^{\mathbb{N}}/\mathcal{B}$ measureable. Now consider the sequence of compositions

$$x(f_n) : [0,1] \to X, \qquad \text{with} \qquad x(k) = x_k \in X.$$

This brings with it a sequence of measures μ_n over $\mathcal{B}$. Note that

$$\mu_n(B) := \lambda(x(f_n) \in B) = \lambda(f_n \in x^{-1}(B)) = \sum_{j \in x^{-1}(B)} a_{nj} = \sum_{j : x_j \in B} a_{nj}.$$

Mathematical Structure

In fact, in the last setup we may as well consider double arrays, if we like, without encountering much difficulties. That is, let $x^{(n)} : \mathbb{N} \to X$ with $x^{(n)}(k) = x_{nk} \in X$.

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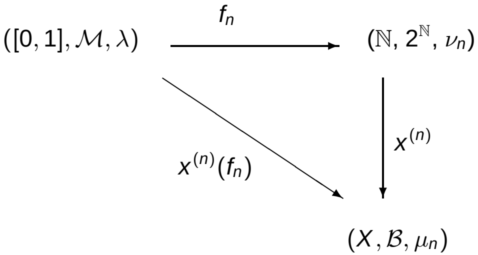

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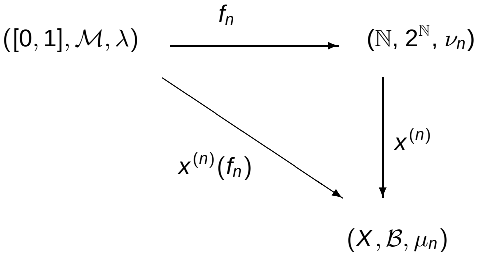

So, we get

$$\mu_n(B) := \lambda(x^{(n)}(f_n) \in B) = \sum_{j : x_{nj} \in B} a_{nj}, \qquad \text{for all } B \in \mathcal{B}.$$

A-Statistical Convergence

So for any nonnegative regular summability matrix $A = [a_{nk}]$ with row sums one, and any double array $x = (x_{nk})$ in X, we get a sequence of measure spaces

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We say $x = (x_{nk})$ is A-statistically convergent to $\alpha \in X$ if for every open set U_α that contains α, we have

$$\lim_{n \to \infty} \mu_n(U_\alpha^c) = 0.$$

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- (iii) What kind of Abelian theory does this spawn?
- (iv) And of course, what kind of Tauberian theory does this spawn?

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All of the above references are concerned with the nature of the convergence field. Of course summability theory goes in two opposite directions — the Abelian side and the Tauberian side —.

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Towards the Abelian direction, A-statistical convergence raises the fundamental issue of whether it can be characterized via a convergent subsequence whose indicies form a set of A-density one. It is not difficult to show that over metric spaces this is possible along the same lines as shown by Fridy. We will have a bit more to say for topological spaces here.

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The answer is easy. Yes. If x_k is convergent to α in X, then for any open set U_α containing α, we can find an N such that $x_k \in U_\alpha$ for all $k > N$. Therefore,

$$\mu_n(U_\alpha^c) = \sum_{k: x_k \notin U_\alpha^c} a_{nk} \leq \sum_{k=0}^{N} a_{nk}.$$

Since $A = [a_{nk}]$ is regular, we see that

$$\lim_{n\to\infty} \mu_n(U_\alpha^c) \leq \lim_{n\to\infty} \sum_{k=0}^{N} a_{nk} = 0.$$

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When X happens to be metrizable, with metric ρ, this notion can be written as follows. For any $\epsilon > 0$ there exists an N so that

$$\lim_{n\to\infty} \sum_{k:\ \rho(x_k,\alpha)>\epsilon} a_{nk} = 0.$$

Density convergence property

Let A be a nonnegative regular summability method and let $x = (x_k)$ be a sequence taking values in a T_2 topological space X. If there exists a set $E \subseteq \mathbb{N}$ such that

$$\delta_A(E) := \lim_{n\to\infty} \sum_{k\in E} a_{nk} = 0,$$

and x is convergent to some α over E^c, then we will say that x has A-density convergence property (DCP(A) for short).

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It is easy to see that if x has the DCP(A) then x is A-statistically convergent to α, where α is its subsequential limit over its E^c.

The issue is whether the converse can hold. This is partially addressed by the following theorem.

Density convergence property

Theorem (DCP(A) vs. A-stat convergence)

Let X be a topological space and let $\alpha \in X$ have a countable base. Then for any nonnegative regular summability matrix A, any A-statistically convergent sequence to α has the DCP(A).

Density convergence property

Theorem (DCP(A) vs. A-stat convergence)

Let X be a topological space and let $\alpha \in X$ have a countable base. Then for any nonnegative regular summability matrix A, any A-statistically convergent sequence to α has the DCP(A).

We are unable to drop the assumption on the countability of the base of α, however one can construct examples outside the countability condition. The general problem seems to be still open over arbitrary T_2 spaces and nonnegative regular matrices A.

DCP is a topological property

The next result shows that the DCP is a topological property.

Theorem

Let X, Y be homeomorphic topological spaces, and let A be any nonnegative regular matrix. If every A-statistically convergent sequence in X has the DCP(A) then every A-statistically convergent sequence in Y also has the DCP(A).

Gap Tauberian condition

Let $\gamma : \mathbb{N} \to \mathbb{N}$ denote an increasing function with $\gamma(0) = 0$. Let

$$G(\gamma) = \{x = (x_k) : x_k \neq x_{k+1} \text{implies there exists } r \in \mathbb{N} \text{ such that } k = \gamma(r)\}$$

Definition

For a nonnegative regular matrix A, we say that $G(\gamma)$ is an A-statistical gap Tauberian condition if $x \in G(\gamma)$ and x is A-statistically convergent to some α together imply that x is convergent.

Topological invariance

Our first result in the Tauberian direction shows topological invariance for statistical gap Tauberian theorems. That is they do not depend on the underlying topological structure at all. They are truely controlled by the summability method used!

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Let A be a nonnegative regular matrix. The following statements are equivalent.

- $G(\gamma)$ *is an A-statistical gap Tauberian condition for real valued sequences.*
- $G(\gamma)$ *is an A-statistical gap Tauberian condition for any Hausdorff topological space valued sequences.*

Topological invariance

An idea of Connor (1993) can now be used to get the following characterizations. For a metric space valued sequence $x = (x_k)$ we say x is strongly A-summable to α if

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk}\,\rho(x_k,\alpha) = 0.$$

Corollary

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Let A be a nonnegative regular matrix. The following statements are equivalent.

- *$G(\gamma)$ is an A-statistical gap Tauberian condition for any T_2 topological space valued sequences.*
- *$G(\gamma)$ is a gap Tauberian condition for A-strong convergence for metric spaces.*

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An idea of Connor (1993) can now be used to get the following characterizations. For a metric space valued sequence $x = (x_k)$ we say x is strongly A-summable to α if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}\,\rho(x_k, \alpha) = 0.$$

Corollary

Let A be a nonnegative regular matrix. The following statements are equivalent.

- *$G(\gamma)$ is an A-statistical gap Tauberian condition for any T_2 topological space valued sequences.*
- *$G(\gamma)$ is a gap Tauberian condition for A-strong convergence for metric spaces.*
- *For all increasing subsequences of $\{n_r\}$ of natural numbers,*

$$\limsup_{n} \sum_{r} \sum_{k\in(\gamma(n_r),\gamma(n_r+1)]} a_{nk} > 0.$$

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So the race is on: find these $\gamma(k)$ for various classical summability methods.

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Theorem

Let $\{k(1), k(2), \cdots\}$ be an increasing sequence of positive integers such that

$$\liminf_i \frac{k(i+1)}{k(i)} > 1, \tag{1}$$

and let x be a sequence in a topological space such that x remains constant over the gaps $(k(i), k(i+1)]$. If x is C_1-statistical convergent to α then x converges to α.

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Here C_1 stands for the Cesàro matrix. In fact, as the following theorem shows, the Cesàro matrix can be replaced by a general nonnegative regular Hausdorff matrix.

Gap conditions: Hausdorff

Theorem

Let H_ϕ be a regular Hausdorff method with a nondecreasing weight function ϕ. Again assume

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holds. If x is a sequence in a topological space such that x remains constant over the gaps $(k(i), k(i+1)]$ and if x is H_ϕ-statistical convergent to α then x converges to α.

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In this theorem we may take $\gamma(t) = ct$ for any constant $c > 1$. The Tauberian condition can be improved if the weight function ϕ of the Hausdorff method has a point of jump.

Gap conditions: Hausdorff with jumps

Theorem

Let H_ϕ be a regular Hausdorff method with a nondecreasing weight function ϕ, having a point of jump at some $r \in (0,1)$. Let $\{k(1), k(2), \cdots\}$ be an increasing sequence of positive integers such that

$$\liminf_i \frac{k(i+1) - k(i)}{\sqrt{k(i)}} > 0. \tag{2}$$

If x is a sequence in a topological space such that x remains constant over the gaps $(k(i), k(i+1)]$ and if x is H_ϕ-statistically convergent to α then x converges to α.

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In this result we may take $\gamma(t) = ct^2$ with $c > 0$. This theorem, in particular, provides a Tauberian theorem for the Euler-statistical convergence. Since the Euler method is also a member of the convolution methods, it is natural to suspect that it may have an analog for the convolution methods. This is indeed the case.

Gap conditions: Convolution methods

Theorem

Let $\{k(1), k(2), \cdots\}$ *be an increasing sequence of positive integers satisfying (2), and let* $A = [a_{nk}]$ *be a regular convolution method with finite variance. If* x *is a sequence in a topological space such that* x *remains constant over the gaps* $(k(i), k(i+1)]$ *and if* x *is A-statistically convergent to* α *then* x *converges to* α.

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We say that $x = (x_k)$ is lacunary statistically convergent to α if for each open set U containing α, we have

$$\lim_{r\to\infty} \frac{|\{k \in (k_{r-1}, k_r] : \ x_k \notin U\}|}{k_r - k_{r-1}} = 0.$$

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The issue is: what is the relationship between the gaps of a lacunary version of a summability method and the gaps of the corresponding Tauberian theorem?

Lacunary vs. gap rates

Proposition

Let X be a T_2 space, and let θ be any lacunary sequence. Then the following statements are equivalent.

- *Every X-valued C_1-statistically convergent sequence is also θ-lacunary statistically convergent.*

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This raises the issue if similar results can be constructed for general A-statistical convergence and their lacunary counterparts. This is also still an open problem.