# Final Exam

The Final Exam is due Monday, May 6, 2003 between 10 am and noon. No late exams will be accepted. You should hand me the exam in person.

Some of the problems require answering questions. Your answers should always be supported by an adequate explanation. I reserve the right to decide whether your explanations are satisfactory or not and base the grade upon my judgment. It is your responsibility to write your answers clearly and in full.

Problems 1–4 are worth the total of 50 points (Problems 1–3 are 10 points each and Problem 4 is 20 points). Problem 5 is worth 15 points on top of that (this is what you call the extra credit).

**Problem 1.** Find the Fourier expansion of

$$f(x) = \begin{cases} x + \pi, & \text{for } -\pi \le x \le 0, \\ -x + \pi, & \text{for } 0 < x \le \pi. \end{cases}$$

Use Dini's test to find the points at which the Fourier series of converges to $f$.

**Problem 2.** Use the method of characteristics to show that the solution of

$$\begin{aligned} &x^3 u_x - u_y = 0, \\ &u(x,0) = 1/(1+x^2), \quad -\infty < x < \infty, \end{aligned}$$

is

$$u(x,y) = (1 - 2x^2 y)/(1 + x^2 - 2x^2 y).$$

Show that the solution is not defined in $y > 1/2x^2$ despite the fact that the data is prescribed for all $x$. Explain why this happens.

**Problem 3.** Solve $u_x + 3u_y = u - 1$ subject to the Cauchy condition $u = x/3$ on $x = \alpha y$, giving reasons for any restrictions that must be placed on $\alpha$.

**Problem 4.** Given PDE $u_t + uu_x = 0$, solve the following two Cauchy problems

$$a) \quad u(x,0) = \begin{cases} 2, & x < 0, \\ 3, & 0 < x, \end{cases} \qquad b) \quad u(x,0) = \begin{cases} 3, & x < 0, \\ 2, & 0 < x. \end{cases}$$

For which of the problems the unique solution satisfying the *principle of causality* is continuous? Which one is the shock solution? What is the speed of propagation of the shock that you obtain in this case from Rankine-Hugoniot condition?

**Problem 5.** Show that if

$$
\begin{aligned}
&u_t + uu_x = 0, \quad t > 0,\\
&u(0,x) = \begin{cases} 0, & x \leq 0 \text{ or } x \geq 1 \\ x(1-x), & 0 \leq x \leq 1, \end{cases}
\end{aligned}
$$

then for $0 \leq \sigma \leq 1$, the characteristics are

$$x - \sigma = \sigma(1-\sigma)t$$

with

$$u = \sigma(1-\sigma).$$

Show also that

$$u^2t^2 + u(1+t-2tx) + x^2 - x = 0,$$

and show that $u$ is continuous for small $t$.

Let $\phi(x,t,\sigma) = x - \sigma - \sigma(1-\sigma)t$. The envelope of the characteristics is found by eliminating $\sigma$ between the equations

$$
\begin{aligned}
&\phi(x,t,\sigma) = 0,\\
&\tfrac{\partial\phi}{\partial\sigma}(x,t,\sigma) = 0;
\end{aligned}
$$

show that this gives the equation for the envelope $4tx = (t+1)^2$ and deduce that a shock forms at $t = 1$, $x = 1$. Verify that the Rankine-Hugoniot condition holds along $x = (t+1)^2/4t$, $t > 1$.