Approximate invariant manifold of the Allen-Cahn flow in two dimensions

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Abstract

In this paper we study transition layers in the solutions to the Allen-Cahn equation in two dimensions. We show that one can construct an approximate invariant manifold to this equation using a version of Lyapunov-Schmidt reduction. Given such manifold one can show that for any straight line segment intersecting the boundary of the domain orthogonally there exists a solution to the Allen-Cahn equation, whose transition layer is located near this segment. The last result has been proven by the author in [12], where the construction of the approximate invariant manifold relied on formal asymptotic expansion.

1 Introduction

In this paper we consider the following elliptic problem:

$$\begin{aligned} &\varepsilon^2\Delta u + f(u) = 0 \qquad \text{in } \Omega, \\ &\partial_n u = 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $f(u) = u(1-u^2)$, $\Omega \in \mathbf{R}^2$ is a bounded domain with smooth boundary, ε is a small parameter and ∂_n denotes the derivative in the direction of the outward normal. Equation (1.1) is know as the Allen-Cahn equation and was introduced in [2] as a model describing the evolution of antiphase boundaries.

The stationary problem (1.1) and its parabolic counterpart have been a subject of an extensive research for many years. In order to describe some of the known results for (1.1) we define the Allen-Cahn functional

$$J(u) = \int_{\Omega} [\frac{\varepsilon^2}{2}|\nabla u|^2 - F(u)], \quad F(u) = -\frac{1}{4}(1-u^2)^2.$$

By $\mathrm{Per}_{\Omega}(A)$ we denote the perimeter of the set $A \subset \Omega$. Intuitively the gradient flow of J, in the limit as $\varepsilon \to 0$, reduces to the gradient flow of Per_{Ω}. It is known that the gradient flow of Per_{Ω} is simply the motion by mean curvature of ∂A. Summarizing: transition layers in the Allen-Cahn flow evolve, as $\varepsilon \to 0$, by their mean curvature. We refer the reader to [16, 5, 7, 8, 9, 17, 10] for more details of this aspect of the problem.

The stationary Allen-Cahn equation was, among others, analyzed in [11]. The authors used Γ-convergence techniques to show that in a neighborhood of a local, isolated minimizer of Per_{Ω} there exists a local minimizer to the functional J. They further used this idea to show the existence of stable solutions for (1.1) in two dimensional, non-convex domains, such as a dumbbell (see also [15]).

In [12] we studied the Allen-Cahn equation in two dimensions and showed that for any smooth, stationary and nondegenerate solution to the mean curvature flow there is a corresponding stationary solution to the Allen-Cahn equation. This result in some sense completes the results described above as it establishes the connection between functionals J and Per_{Ω} on the level of their critical points.

Throughout this paper we assume that a curve $\gamma \in \Omega$, our candidate for an interface, is such that:

(i) the curvature of γ is 0 (γ is a straight line segment);

(ii) γ intersects $\partial\Omega$ at exactly two points γ_0, γ_1 and at those points $\gamma \perp \partial\Omega$;

(iii) γ is nondegenerate in the sense described below (see (1.6) to follow).

In [12] we proved the following:

Theorem 1.1 *Let U be the unique heteroclinic solution to*

$$\begin{aligned} &U_{\eta\eta} + f(U) = 0, \qquad && -\infty < \eta < \infty, \\ &U(\pm\infty) = \pm 1, U(0) = 0. && \end{aligned} \tag{1.2}$$

Let $d(\gamma; x, y)$ denote the signed distance of a point $(x, y) \in \Omega$ to the straight line that contains γ. For each sufficiently small ε there exists a solution u^{ε}

to (1.1) such that

$$\|u^\varepsilon(x,y) - U(d(\gamma;x,y)/\varepsilon)\|_{C^0(\Omega)} \leq C\varepsilon, \tag{1.3}$$

where $C > 0$ is independent on ε.

Recently Pacard and Ritoré [14] have proven a generalization of the existence result presented here. Namely, they showed that any nondegenerate minimal hypersurfaces in $(n+1)$, $n \geq 1$, Riemanian manifold is a nodal set of a solution to the Allen-Cahn equation. The key idea in their approach is to invert the linearized operator in carefully chosen weighted spaces.

In [12] we also analyzed the Morse index of the solution described in Theorem 1.1. More precisely we study the following eigenvalue problem:

$$\begin{aligned} &\varepsilon^2\Delta V + f'(u^\varepsilon)V = -\Lambda V \qquad && \text{in } \Omega, \\ &\partial_n V = 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

The Morse index of u^ε is simply the number of negative eigenvalues of (1.4).

To state our result we need to define a geometric eigenvalue problem that, as we will see, plays an important role in our considerations. Let $\kappa_{\partial\Omega}(\gamma_i)$, $i = 0, 1$ be the curvatures of $\partial\Omega$ at the points of intersection with γ. Consider the following eigenvalue problem

$$\begin{aligned} -\theta_{ss} &= \lambda\theta, && 0 < s < |\gamma|, \\ \theta\kappa_{\partial\Omega}(\gamma_0) + \theta_s &= 0, && s = 0, \\ -\theta\kappa_{\partial\Omega}(\gamma_1) + \theta_s &= 0, && s = |\gamma|, \end{aligned} \tag{1.5}$$

where γ is parameterized by arclength in such a way that s increases from γ_0 to γ_1 and $\partial\Omega$ is oriented counterclockwise from γ_0 to γ_1. We say that γ is non-degenerate if (1.5) does not have a zero eigenvalue. This is equivalent to the following condition:

$$\kappa_{\partial\Omega}(\gamma_0) + \kappa_{\partial\Omega}(\gamma_1) - \kappa_{\partial\Omega}(\gamma_0)\kappa_{\partial\Omega}(\gamma_1)|\gamma| \neq 0. \tag{1.6}$$

We can now state our second theorem.

Theorem 1.2 *The Morse index of the solution to (1.1) described in Theorem 1.1 equals the number of negative eigenvalues of (1.5). Moreover for any $k^* > 0$ there exists ε_{k^*} such that, for all $\varepsilon \in (0, \varepsilon_{k^*}]$, if $\{\Lambda_k\}_{k=1,\dots,k^*}$ are the first k^* eigenvalues of the linearized problem (1.4) then $\Lambda_k = \lambda_k\varepsilon^2 + o(\varepsilon^2)$, where $\{\lambda_k\}_{k=1,\dots,k^*}$ are the first k^* eigenvalues of (1.5).*

Remark 1.1 *By an explicit calculation one can show that (1.5) has at most two negative eigenvalues. For example if both curvatures are negative then the spectrum is positive. In this case γ is a local minimizer of the perimeter and this situation was treated in [11]. If both curvatures are positive then:*

- *If $\frac{1}{\kappa_{\partial\Omega}(\gamma_0)} + \frac{1}{\kappa_{\partial\Omega}(\gamma_1)} > |\gamma|$ then (1.5) has one negative eigenvalue (short axis case).*
- *If $\frac{1}{\kappa_{\partial\Omega}(\gamma_0)} + \frac{1}{\kappa_{\partial\Omega}(\gamma_1)} < |\gamma|$ then (1.5) has two negative eigenvalues (long axis case).*

Obviously, except a degenerate case, any combination of the curvatures gives rise to one of the three cases described above.

Some key ideas in [12] were motivated by [1]. In this paper the authors considered the dynamics of the mass conserving Allen-Cahn equation. Starting from a one parameter family of approximate interfaces with constant mean curvature intersecting the boundary they were able to construct an approximate invariant manifold to the parabolic PDE consisting of small drops moving along the boundary. Their construction relies on the fact that the interfaces for the mass conserving Allen-Cahn equation evolve by volume-preserving mean curvature flow [4].

In [12] the analogous step in the proof of the existence is carried out by using the formal asymptotic expansion technique. In the present paper we outline another method of constructing the approximate invariant manifold. It relies on a version of Lyapunov-Schmidt reduction which, in the context of the Allen-Cahn equation, we have developed in [6] where we show the existence of phantom interfaces in 2 dimensions (see also [3], [13]). (Phantom are solutions to (1.1) with multiple transition layers collapsing onto each other as $\varepsilon \to 0$).

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2 Approximate manifold equation

2.1 Change of variables

We can assume that after rotation and scaling $\gamma = \{(y_1, y_2) \mid y_1 = 0, 0 < y_2 < 1\}$. Let $\nu > 0$ be a fixed number and let $\Omega_\nu = \{|y| < \nu\} \cap \Omega$. We introduce new variables in Ω_ν as follows: assume that $\partial\Omega_\nu \cap \partial\Omega$ is expressed near $y_2 = 0$ ($y_2 = 1$ respectively) as a graph of smooth function $y_2 = g_0(y_1)$

($y_2 = g_1(y_1)$ respectively). Let η be a smooth cut off function such that $\eta(s) = 1$, for $|s| < 1$ and $\eta(s) = 0$ for $|s| > 2$. Let $\sigma > 0$ be a small fixed number. Set

$$\zeta = y_2 - \eta(y_2/\sigma)g_0(y_1) - \eta((1-y_2)/\sigma)[g_1(y_1) - 1]$$

We fix ν, σ such that the change of variables $(y_1, y_2) \mapsto (y_1, \zeta)$ is a smooth map from Ω_ν to the cylinder $C_\nu = \{|y_1| < \nu, 0 < \zeta < 1\}$. We also introduce stretched variables $x = y_1/\varepsilon$, $z = \zeta/\varepsilon$ in the stretched cylinder $C_{\nu/\varepsilon}$. For future references we observe that in the new variables:

$$\varepsilon^2 \Delta_y = \Delta_{x,z} + B$$

where B is a second order differential operator with coefficients of order $O(\varepsilon|x|) + O(\varepsilon)$. In particular it is a small perturbation of Δ provided that ν is sufficiently small. We also have near $z = 0$:

$$\varepsilon \partial_n = \frac{-1 + |g_0'(\varepsilon x)|^2}{(1 + |g_0'(\varepsilon x)|^2)^{1/2}} \partial_z + \frac{g_0'(\varepsilon x)}{(1 + |g_0'(\varepsilon x)|^2)^{1/2}} \partial_x,$$

with a similar formula near $z = 1/\varepsilon$. We see that $\varepsilon \partial_n = \pm \partial_z + b$, where the coefficients of the boundary operator b are of order $O(\varepsilon|x|)$. Let $\phi \in H^2(0,1)$ be given and consider a function of the form $v = v(x - \frac{1}{\varepsilon}\phi(\varepsilon z))$. Then we have, say at $z = 0$:

$$\begin{aligned} -\partial_z v + bv \;&= \{[-1 + |g_0'(\phi(0))|^2]\phi'(0) + g_0'(\phi(0))\} v' \\ &+ O(\varepsilon|x - \tfrac{1}{\varepsilon}\phi(0)|) v'. \end{aligned}$$

In the sequel we will assume that ϕ is such that

$$[-1 + |g_j'(\phi(\zeta))|^2]\phi'(\zeta) + g_j'(\phi(\zeta)) = 0, \quad j, \zeta = 0, 1 \tag{2.1}$$

so that for functions depending on $x - \frac{1}{\varepsilon}\phi(\varepsilon z)$ we have that $\pm \partial_z v + bv = O(\varepsilon|x - \frac{1}{\varepsilon}\phi|)v'$. We observe that linearizing (2.1) we obtain the boundary conditions in (1.5). We also notice that any function ϕ that satisfies (2.1) can be uniquely represented as $\phi(\zeta) = \phi_1 + m_1\zeta + m_0$, where ϕ_1 satisfies the linearized boundary conditions. This is thanks to the non-degeneracy condition.

2.2 Statement of the problem

Recall that $U = U(\eta)$ is the heteroclinic solution to (1.2) such that $U(\pm\infty) = \pm 1$.

Let Θ_k, λ_k denote the eigenfunctions and eigenvalues of (1.5) and let ϕ satisfying (2.1) be given. We set

$$Z_k(x,z) = U'(x - \frac{1}{\varepsilon}\phi(\varepsilon z))\Theta_k(\varepsilon z).$$

Our goal is to construct a solution to the following problem:

$$\begin{aligned} &\varepsilon^2 \Delta u + f(u) = \textstyle\sum_{k=1}^{K_\varepsilon} c_z Z_k + g^\varepsilon \qquad \text{in } \Omega, \\ &\partial_n u = h^\varepsilon \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

The unknowns here are: $u = u(y, \phi)$, $\mathbf{c} = \{c_k\}$, $y = (y_1, y_2) \in \Omega$. In order to construct solution to (2.2) we will consider inner, outer and boundary layer expansion of u. Functions $g^\varepsilon = g^\varepsilon(y)$ and $h^\varepsilon = h^\varepsilon(y)$ come from 'gluing' approximate solutions in those regions. The above equation is to be valid for $y \in \Omega$, $\|\phi\|_{H^2} \le M\varepsilon$, $M > 0$ large. We take $K_\varepsilon = [1/\varepsilon]$. We speak of solution to (2.2) up to exponentially small order if $g^\varepsilon = O(e^{-c/\varepsilon})$ and $h^\varepsilon = O(e^{-c/\varepsilon})$.

We will first consider the approximate manifold problem in Ω_ν. Switching to stretched variables we get the following 'interior' problem

$$\begin{aligned} &\Delta u + Bu + f(u) = \textstyle\sum_{k=1}^{K_\varepsilon} c_z Z_k + g^\varepsilon \qquad \text{in } C_{\nu/\varepsilon}, \\ &\pm\partial_z u + bu = h^\varepsilon \qquad \text{on for } z = 0, 1/\varepsilon. \end{aligned} \tag{2.3}$$

In the rest of this note we concentrate on solving problem (2.3). It is convenient to formulate this problem in the infinite strip $\mathcal{S} = \{0 < z < 1/\varepsilon\}$ instead of $C_{\nu/\varepsilon}$ since the solution to the original problem can be obtained by taking an appropriate cutoff of the solution in $\mathcal{S}$. We will look for the solution in the form:

$$u(x,z) = w(x,z) + \psi(x,z), \quad w(x,z) = U(x - \frac{1}{\varepsilon}\phi(\varepsilon z)).$$

We let

$$L\psi = \Delta\psi + B\psi + f'(w)\psi.$$

We will introduce notation

$$N(\psi) = f'(w)\psi + f(w) - f(w+\psi), \quad R(w) = -\Delta w - Bw - f(w)$$

Our problem now is to solve

$$\begin{aligned} &L\psi = R(w) + \textstyle\sum_{k=1}^{K_\varepsilon} c_k Z_k + N(\psi) \qquad \text{in } \mathcal{S}, \\ &(\pm\partial_z + b)\psi = -(\pm\partial_z + b)w \qquad \text{on } \partial\mathcal{S}. \end{aligned}$$

Notice that passing from the solution defined in $\mathcal{S}$ to the one defined in Ω_ν we need to introduce a cut-off version of u, which in turn leads to terms g^ε and h^ε in (2.2), as pointed out earlier.

2.3 Solving the linear problem

We will state without a proof a Proposition that allows to solve the linearized problem. We refer the reader to [6] for details.

Proposition 2.1 *Consider the following problem*

$$\begin{aligned} &Lv = h + \textstyle\sum_{k=1}^{K_\varepsilon} c_k Z_k \qquad \textit{in } \mathcal{S}, \\ &(\pm\partial_z + b)\psi = g \qquad \textit{on } \partial\mathcal{S} \\ &\textstyle\int_{\mathcal{S}} v Z_k = 0, \quad k = 1, \dots, K_\varepsilon. \end{aligned} \tag{2.4}$$

Assume that $h \in L^2(\mathcal{S})$, $g \in L^2(\partial\mathcal{S})$.

There exists a unique solution to (2.4) in addition satisfying estimate

$$\|v\|_{H^2(\mathcal{S})} \leq C \left[\|h\|_{L^2(\mathcal{S})} + \left(\sum_{k=1}^{K_\varepsilon} c_k^2 \|Z_k\|^2_{L^2(\mathcal{S})} \right)^{1/2} + \|g\|_{L^2(\partial\mathcal{S})} \right]$$

where constants c_k are determined from the following expression:

$$c_k \|Z_k\|^2_{L^2(\mathcal{S})} = \int_{\mathcal{S}} (Lv - h) Z_k$$

We observe here that $\|Z_k\|^2_{L^2(\mathcal{S})} = \varepsilon^{-1}\alpha_0$, where $\alpha_0 = \int |U'|^2$.

2.4 Boundary layer expansion

We further write $\psi = \psi_0 + \psi_1$ where ψ_0 is of order $O(\varepsilon)$ and corresponds to a boundary layer term, while ψ_1 is of order $O(\varepsilon^{3/2})$. In this section we will use Proposition 2.1 to solve the following problem for ψ_0:

$$\begin{aligned} &L\psi_0 = \textstyle\sum_{k=1}^{K_\varepsilon} c_{0k} Z_k, \qquad \text{in } \mathcal{S} \\ &(\pm\partial_z + b)\psi_0 = -(\pm\partial_z + b)w, \qquad \text{on } \partial\mathcal{S} \\ &\textstyle\int_{\mathcal{S}} \psi_0 Z_k = 0, \qquad k = 1, \dots, K_\varepsilon \end{aligned} \tag{2.5}$$

In fact the existence of a unique solution to (2.5) is guaranteed from by Proposition 2.1. We we will compute c_{0k}'s in order to obtain the estimate for ψ_0. We have:

$$\begin{aligned} c_{0k}\varepsilon^{-1}\alpha_0 \;&= \textstyle\int_{\mathcal{S}} (B\psi_0) Z_k + \int_{\mathcal{S}} [\Delta Z_k + f'(w) Z_k]\psi_0 \\ &\quad + \textstyle\int_{\partial\mathcal{S}} [Z_k \partial_z \psi_0 - \psi_0 \partial_z Z_k] \end{aligned}$$

Straightforward calculations yield the following estimate:

$$|c_{0k}| \leq C\varepsilon^{3/2}\|\psi_0\|_{H^2(\mathcal{S})} + C\varepsilon^3 \tag{2.6}$$

where we have taken into account $\|\phi\|_{H^2(0,1)} \leq M\varepsilon$. Thus

$$\varepsilon^{-1}\sum_{k=1}^{K_\varepsilon} c_{0k}^2\alpha_0 \leq C\varepsilon\|\psi_0\|^2_{H^2(\mathcal{S})} + C\varepsilon^4$$

which together with the H^2 estimate in Proposition 2.1 gives

$$\|\psi_0\|_{H^2(\mathcal{S})} \leq C\varepsilon \tag{2.7}$$

2.5 Solving the problem with homogeneous boundary conditions

We are going to solve the following problem for ψ_1:

$$\begin{aligned} &L\psi_1 = R(w) + \textstyle\sum_{k=1}^{K_\varepsilon} c_{1k}Z_k + N(\psi_0+\psi_1) \qquad \text{in } \mathcal{S}, \\ &(\pm\partial_z + b)\psi_1 = 0 \qquad \text{on } \partial\mathcal{S}. \end{aligned} \tag{2.8}$$

Observe that the term $\sum_{k=1}^{K_\varepsilon} c_{0k}Z_k$ coming from (2.5) and $\sum_{k=1}^{K_\varepsilon} c_{1k}Z_k$ will at the end be combined so that we get the solution to (2.2) with $c_k = c_{0k} + c_{1k}$.

We will use Proposition 2.1 to set up a fixed point argument to solve (2.8). For that we consider a ball $B_{K\varepsilon^{3/2}}$ in $H^2(\mathcal{S})$, with large K to be chosen. Assume that $v \in B_{K\varepsilon^{3/2}}$. It suffices to show that if ψ_1 is a solution to

$$\begin{aligned} &L\psi_1 = R(w) + \textstyle\sum_{k=1}^{K_\varepsilon} c_{1k}Z_k + N(\psi_0+v) \qquad \text{in } \mathcal{S}, \\ &(\pm\partial_z + b)\psi_1 = 0 \qquad \text{on } \partial\mathcal{S}. \end{aligned}$$

then $\psi_1 \in B_{K\varepsilon^{3/2}}$. Using Proposition 2.1 we know that ψ_1 exists. Furthermore by the H^2 estimate we get:

$$\|\psi_1\|_{H^2(\mathcal{S})} \leq C\left[\|R(w)\|_{L^2(\mathcal{S})} + \left(\sum_{k=1}^{K_\varepsilon}\varepsilon^{-1}\alpha_0 c_{1k}^2\right)^{1/2} + \|N(\psi_0+v)\|_{L^2(\mathcal{S})}\right]$$

Firstly, we have:

$$\|R(w)\|_{L(\mathcal{S})} \leq C\varepsilon^{3/2} \tag{2.9}$$

Deriving (2.9) is rather straightforward after making use of the explicit form of w, the operator B and $\|\phi\|_{H^2(0,1)} \leq M\varepsilon$.

Secondly, using the quadratic nature on $N(\psi_0 + v)$, Sobolev embedding and estimate (2.7) we get:

$$\|N(\psi_0 + v)\|_{L^2(\mathcal{S})} \leq C(\|\psi_0\|_{L^2(\mathcal{S})} + \|v\|_{L^2(\mathcal{S})}) \leq K\varepsilon^3 + C\varepsilon^2.$$

Finally, we multiply $L\psi_1$ be Z_k, $k = 1, \ldots, K_\varepsilon$ to evaluate c_{1k}'s. The estimate we get is similar to (2.6) since, except for the nonlinear term we are dealing with the expression of the same type:

$$|c_{1k}| \leq C\varepsilon^{3/2}\|\psi_1\|_{H^2(\mathcal{S})} + C\varepsilon^{5/2}$$

hence

$$\varepsilon^{-1}\sum_{k=1}^{K_\varepsilon} c_{1k}^2\alpha_0 \leq C\varepsilon\|\psi_0\|^2_{H^2(\mathcal{S})} + C\varepsilon^3$$

Combining all those estimates we clearly get that map $v \mapsto \psi_1$ leaves the ball $B_{K\varepsilon^{3/2}}$ invariant provided that that ε is taken sufficiently small and K is taken sufficiently large.

We now define u, the solution to (2.2) by:

$$u = [U(x - \frac{1}{\varepsilon}\phi(\varepsilon z)) + \psi_0 + \psi_1]\chi_\nu + U(\frac{y_1}{\varepsilon})(1 - \chi_\nu)$$

where χ_ν is a cutoff function supported in $\Omega_{2\nu}$ and equal to 1 in Ω_ν. This ends the construction of the approximate invariant manifold.

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