Practice problems

Problem 1. Show that the following functions form an orthogonal set

(a) $\sin(2n+1)x$, on the interval $[0, \pi/2]$, $n = 0, 1, 2, \ldots$.

(b) $sin((n + \frac{1}{2})x$ on the interval $[-\pi, \pi]$, $n = 0, 1, 2, \ldots$.

Problem 2. Show that a square integrable function f satisfies the Dini's test, i.e.

$$\int_{-\pi}^{\pi} \frac{|f(x+\tau) - f(x)|}{|\tau|}\, d\tau < \infty$$

if and only if it satisfies

$$\int_{-\pi}^{\pi} \frac{|f(x+\tau) - f(x)|}{|\sin\tau|}\, d\tau < \infty.$$

Hint: Show that if $|\tau| \leq 1$ it holds

$$\frac{|\sin\tau|}{|\tau|} \leq C_1, \quad \text{and} \frac{|\tau|}{|\sin\tau|} \leq C_2$$

for certain constants C_1, C_2.

Problem 3. Give an example of a function which is continuous but *not* Hölder continuous at $x = 0$. Can you find an example of a function whose Fourier series converges at all points of the interval $(-\pi, \pi)$ which is not Hölder continuous at at least one point. Hint: consider functions of the form $f(x) = \frac{1}{|\log x|^p}$, $p > 0$.

Problem 4. Use the method of characteristics to solve the following Cauchy problems.

(a) $2u_x - 5u_y = 4$, $u(x, 0) = x$.

(b) $u_x + 3u_y = u + 2$, $u(0, y) = y$.

(c) $u_x + u_y = 1 - u$, subject to the Cauchy condition $u = x$ on $y = 2x$.

(d) $u_x - 2u_y = u - 1$, subject to the Cauchy condition $u = 2y$ on $x = ky$, giving reasons for any restrictions that must be placed on k.

(e) $yu_x + x^3u_y = x^3y$, subject to the Cauchy condition $u = x^4$ on $y = x^2$.

(f) $u_x + 2u_y = u$ given that $u(x, 0) = 3x$, $x \geq 0$ and $u(0, y) = \sin y$, $y \geq y$.