Lifts and generalized vertices for Brauer characters of solvable groups

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May 23, 2010

OSU-Denison Conference 2010

(Joint work with J. P. Cossey – University of Akron)

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- When G is p-solvable, the Fong-Swan theorem implies that φ has a lift.

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- J. P. Cossey has initiated the study of all lifts of φ. For example, when $|G|$ is odd, he has shown that the number of lifts of φ can be bounded in terms of a vertex for φ.
- We will show that the oddness hypothesis in Cossey's results can be removed in certain cases.

Vertices

- In a p-solvable group G, we say Q is a *vertex* for $\varphi \in \mathrm{IBr}(G)$ if there is a subgroup U so that φ is induced from a p-Brauer character of U having p'-degree and Q is a Sylow p-subgroup of U.

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- It is known that all of the vertices for φ are conjugate in G.
- Cossey showed that if $|G|$ is odd and Q is a vertex for φ, then the number of lifts of φ is at most $|Q : Q'|$.

Vertices

We now remove the hypothesis that $|G|$ is odd.

However, we do need to add some hypotheses:

1. G is p-solvable
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3. Q is abelian

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Theorem

Let G be a p-solvable group and let p be an odd prime. If $\varphi \in \mathrm{IBr}(G)$ has abelian vertex Q, then the number of lifts of φ is at most $|Q|$.

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- A character $\chi \in \mathrm{Irr}(G)$ is p-factored if $\chi = \alpha\beta$ where α is p-special and β is p'-special.
- Let $\chi \in \mathrm{Irr}(G)$. Then (Q, δ) is a *generalized vertex* for χ if there is a subgroup U with a p-factored character $\psi \in \mathrm{Irr}(U)$ and Sylow p-subgroup Q of U so that $\psi^G = \chi$ and δ is the restriction to Q of the p-special factor of ψ.

Generalized vertices

Since any primitive irreducible character of a p-solvable group is p-factored and p-special characters restrict irreducibly to a Sylow p-subgroup, all characters have generalized vertices.

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1. *δ is linear*
2. *all generalized vertices for χ are conjugate to (Q, δ).*

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To get δ linear, we use a recent theorem of Navarro:

Theorem

(Navarro) Let G be a p-solvable group for odd prime p. Let $\chi \in \mathrm{Irr}(G)$ be p-special. If $\chi(1) > 1$, then χ^o is not in $\mathrm{IBr}(G)$.

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Note: this theorem is not true if $p = 2$.

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As a corollary to Navarro's result, we obtain the following:

Corollary

Let G be a p-solvable group where p is an odd prime. If $\chi \in \mathrm{Irr}(G)$ satisfies $\chi^o \in \mathrm{IBr}(G)$ and has generalized vertex (Q, δ), then δ is linear.

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If $p = 2$, this corollary is not true. In $\mathrm{GL}_2(3)$, there is a counterexample.

Generalized vertices

We now prove:

Theorem

Let G be a p-solvable group and p an odd prime. If $\chi \in \mathrm{Irr}(G)$ with $\chi^o \in \mathrm{IBr}(G)$, then all the generalized vertices for χ are conjugate.

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When $p = 2$, the theorem is not true.

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Theorem

Assume that G is a p-solvable group and p is an odd prime. Suppose that $\varphi \in \mathrm{IBr}(G)$ has vertex subgroup Q that is abelian, and let $\delta \in \mathrm{Irr}(Q)$. Then $|L_\varphi(Q,\delta)| \leq |\mathrm{N}_G(Q) : \mathrm{N}_G(Q,\delta)|$.

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Applying the count for the generalized vertices: $\sum_{i=1}^k |L_\varphi(Q, \delta_i)| \leq \sum_{i=1}^k |\mathrm{N}_G(Q) : \mathrm{N}_G(Q, \delta_i)|$.

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Finally, counting the sizes of the orbits of $\mathrm{N}_G(Q)$ on the linear characters of Q, we obtain $\sum_{i=1}^{k} |\mathrm{N}_G(Q) : \mathrm{N}_G(Q, \delta_i)| = |Q|$.

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Combining: $|L_\varphi| \leq |Q|$. (As desired.)

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By the inductive hypothesis, we know that $|L_\zeta(Q, \delta)| \leq |N_I(Q) : N_I(Q, \delta)|$.

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In this particular case, we can show that $N_I(Q, \delta) = N_G(Q, \delta)$. The result follows if we can show that the number of Brauer characters in I with vertex Q that induce φ is at most $|N_G(Q) : N_I(Q)|$.

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When Q is abelian, we can show that this occurs in our situation.

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Let G be a p-solvable group. Suppose $\varphi \in \mathrm{IBr}(G)$ has vertex Q. Suppose $Q \leq I \leq G$. Is it true that the number of characters in $\mathrm{IBr}(I)$ with vertex Q that induce φ is at most $|\mathrm{N}_G(Q) : \mathrm{N}_I(Q)|$?

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We have not been able to settle this question at this time.

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Also, when p is odd, we can prove that if G is a minimal counterexample, then I is a maximal subgroup, $|G : I|$ is a power of 2, and φ restricts homogeneously to every normal subgroup of G contained in I. Furthermore, writing N for the core of I in G and M for a normal subgroup of G so that M/N is a chief factor of G, if α is the irreducible constituent of φ_N, then α^M has a unique irreducible constituent.