Bounding an index by the largest character degree of a solvable group

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Introduction

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- We write $b(G)$ for the largest irreducible character degree of G.
  I.e., $b(G) = \max\{a \mid a \in \mathrm{cd}(G)\}$.
- We investigate the relationship between $|G : \mathbf{O}_p(G)|_p$ and $b(G)$.

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- Motivated by these results, Benjamin proved that if G is p-solvable and $b(G) < p^2$, then $|G : \mathbf{O}_p(G)|_p < p^2$.

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- In Theorem 12.32, he proves if $b(G) < p^{3/2}$, then $|G : \mathbf{O}_p(G)|_p < p^2$.
- Motivated by these results, Benjamin proved that if G is p-solvable and $b(G) < p^2$, then $|G : \mathbf{O}_p(G)|_p < p^2$.
- Benjamin also proved that if G is solvable, then $|G : \mathbf{O}_p(G)|_p \leq (b(G))^2$. If in addition $|G|$ is odd, then $|G : \mathbf{O}_p(G)|_p \leq b(G)$.

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- Qian and Shi also proved that if G is any group, then $|G : \mathbf{O}_p(G)|_p \leq (b(G))^2$. If G has an abelian Sylow p-subgroup, then $|G : \mathbf{O}_p(G)|_p \leq b(G)$.

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- Corollary:
  1. If G is solvable and $b(G) < p^p$, then $|G : \mathbf{O}_p(G)|_p \leq b(G)$.
  2. If G is solvable and a Sylow p-subgroup has nilpotence class at most p, then $|G : \mathbf{O}_p(G)|_p \leq b(G)$.

Our results

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Theorem

Let G be a p-solvable group and let p be an odd prime that is not a Mersenne prime. Then $|G : \mathbf{O}_p(G)|_p \leq b(G)$.

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Theorem

Let G be a p-solvable group and suppose that G does not have a section isomorphic to $z_p \wr Z_p$. Then $|G : \mathbf{O}_p(G)_p \leq b(G)$.

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These theorems are not true if we remove the hypothesis that either p is not a Mersenne prime or have a section that is isomophic to $Z_p \wr Z_p$.

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In particular, when p is a Mersenne prime, there exists a solvable group G with $b(G) = p^p$ and $|G : \mathbf{O}_p(G)|_p = p^{p+1}$.

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In particular, when p is a Mersenne prime, there exists a solvable group G with $b(G) = p^p$ and $|G : \mathbf{O}_p(G)|_p = p^{p+1}$.

When f is a Fermat prime, there exists a group G with $b(G) = 2^{2f}$ and $|G : \mathbf{O}_2(G)|_2 = 2^{2f+1}$.

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Note that if $b(G) = p^p$, then $(b(G)^p/p)^{1/(p-1)} = p^{p+1}$. Thus, our first example shows that this bound is best possible when G is p-solvable.

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We have not worked to see if we can remove the p-solvable hypothesis.

Solvable groups

We now outline the arguments in our proofs. We begin by looking at how to prove the results when G is solvable.

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This first lemma is essentially a known result.

Lemma

Let P be a p-group, and assume that P acts faithfully and coprimely on an abelian group V. Assume one of the following conditions:

1. *p is odd and not a Mersenne prime.*
2. *$Z_p \wr Z_p$ is not a section of P.*

Then P has a regular orbit on V.

Solvable groups

Theorem

Let G be a solvable group. Assume either p is an odd prime that is not a Mersenne prime or that G does not have a section isomorphic to $Z_p \wr Z_p$. Then $|G : \mathbf{O}_p(G)|_p \leq b(G)$.

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Outline of proof: Take G to be a counterexample with $|G|$ minimal. We may assume $\mathbf{O}_p(G) = 1$.

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Outline of proof: Take G to be a counterexample with $|G|$ minimal. We may assume $\mathbf{O}_p(G) = 1$.

Let P be Sylow p-subgroup of G and F the Fitting subgroup of G. Note that p does not divide $|F|$. Using the inductive hypothesis, we may assume $G = PF$. Now, P acts faithfully and coprimely on $F/\Phi(F)$.

Solvable groups

We now use the Lemma to see that P has a regular orbit in its action on $\mathrm{Irr}(F/\Phi(F))$. This gives a linear character $\lambda \in \mathrm{Irr}(F)$ whose stabilizer is F. In particular, $\lambda^G \in \mathrm{Irr}(G)$, and so,

$$b(G) \geq \lambda^G(1) = |G : F| = |P| = |G|_p,$$

contradicting the choice of G.

Solvable groups

To prove the general inequality, we need the following result that was proved by Isaacs.

Theorem (Isaacs)

Let P be a p-group that acts faithfully and coprimely on a group V. Then there exists an element $v \in V$ so that $|\mathbf{C}_P(v)| \leq (|P|/p)^{1/p}$.

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Recall our theorem:

Theorem

Let G be a p-solvable group. Then $|G : \mathbf{O}_p(G)|_p \leq (b(G)^p/p)^{1/(p-1)}$.

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We work by induction on $|G|$. By the inductive hypothesis, we may assume that $\mathbf{O}_p(G) = 1$. Let P be a Sylow p-subgroup, and let F be the Fitting subgroup of G. By the inductive hypothesis, $G = PF$.

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We work by induction on $|G|$. By the inductive hypothesis, we may assume that $\mathbf{O}_p(G) = 1$. Let P be a Sylow p-subgroup, and let F be the Fitting subgroup of G. By the inductive hypothesis, $G = PF$.

Now, P acts faithfully and coprimely on $F/\Phi(F)$. We use Isaacs' theorem to find $\lambda \in \mathrm{Irr}(F/\Phi(F))$ so that $|\mathbf{C}_P(\lambda)| \leq (|P|/p)^{1/p}$.

Solvable groups

If T is the stabilizer of λ in G, then $T = F\mathbf{C}_P(\lambda)$, and so

$$|G : T| = |P : \mathbf{C}_P(\lambda)| \geq \frac{|P|}{\left(\frac{|P|}{p}\right)^{1/p}} = \left(|P|^{p-1}p\right)^{1/p}.$$

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This implies that $\left(|P|^{p-1}p\right)^{1/p} \leq b(G)$, and we conclude that $|P| \leq (b(G)^p/p)^{1/(p-1)}$, as desired.

p-solvable groups

We now discuss how to prove our results when G is p-solvable. We first need the following consequence of Gluck's regular orbit theorem which was proved by Dolfi.

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Lemma (Dolfi)

Let p be a prime an odd prime, and suppose that P is a p-group that is a permutation group on Ω. Then there exists a set $\Delta \subseteq \Omega$ so that $P_\Delta = 1$.

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We also need a result of Moretó and Tiep.

Lemma (Moretó and Tiep)

Let A act faithfully and coprimely on a nonabelian simple group S. Then A has at least 2 regular orbits on $\mathrm{Irr}(S)$.

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Lemma

Let S be a nonabelian simple group, and let p be a prime that does not divide $|S|$. Suppose $V = S_1 \times \cdots \times S_n$ where $S_i \cong S$. Assume P is a p-group that acts faithfully on V via automorphisms, and assume the action of P transitively permutes the S_i's. If G is the semi-direct product of P acting on V, then there exists $\theta \in \mathrm{Irr}(V)$ so that θ^G is irreducible.

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We work by induction on $|G|$, and we observe that all sections of G satisfy the inductive hypothesis.

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We work by induction on $|G|$, and we observe that all sections of G satisfy the inductive hypothesis.

Using the inductive hypothesis, we may show that $\mathbf{O}_p(G) = 1$ and $\Phi(G) = 1$.

p-solvable groups

Idea of proof of main theorems when G is p-solvable:

We work by induction on $|G|$, and we observe that all sections of G satisfy the inductive hypothesis.

Using the inductive hypothesis, we may show that $\mathbf{O}_p(G) = 1$ and $\Phi(G) = 1$.

Let F be the Fitting subgroup of G and let P be a Sylow p-subgroup. If all minimal normal subgroups of G are abelian, then use the inductive hypothesis to show that we may assume $G = PF$ and then the result holds via the solvable case.

p-solvable groups

Let V be a minimal normal subgroup of G that is not abelian. Then $V = S_1 \times \cdots \times S_n$ where $S_i \cong S$ and S is a nonabelian simple group with $(|S|, p) = 1$.

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Let V be a minimal normal subgroup of G that is not abelian. Then $V = S_1 \times \cdots \times S_n$ where $S_i \cong S$ and S is a nonabelian simple group with $(|S|, p) = 1$.

Use the inductive hypothesis to assume that $G = H = V\mathbf{C}_G(V)P$.

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Use the inductive hypothesis to assume that $G = H = V\mathbf{C}_G(V)P$.

By the Lemma, we can find $\theta \in \mathrm{Irr}(V)$ so that $\mathbf{C}_P(\theta) = \mathbf{C}_P(V)$.

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Use the inductive hypothesis to assume that $G = H = V\mathbf{C}_G(V)P$.

By the Lemma, we can find $\theta \in \mathrm{Irr}(V)$ so that $\mathbf{C}_P(\theta) = \mathbf{C}_P(V)$.

Let $\gamma \in \mathrm{Irr}(\mathbf{C}_G(V))$ so that $\gamma(1) = b(\mathbf{C}_P(V))$.

p-solvable groups

This implies that $(\theta \times \gamma)^G$ is irreducible, and hence, $b(G) > |P : \mathbf{C}_P(V)| b(\mathbf{C}_G(V))$.

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Using the inductive hypothesis in $\mathbf{C}_G(V)$, we obtain the desired conclusions.