A Dual-Primal FETI method for incompressible Stokes equations

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Abstract

In this paper, a dual-primal FETI method is developed for incompressible Stokes equations approximated by mixed finite elements with discontinuous pressures. The domain of the problem is decomposed into nonoverlapping subdomains, and the continuity of the velocity across the subdomain interface is enforced by introducing Lagrange multipliers. By a Schur complement procedure, the solution of an indefinite Stokes problem is reduced to solving a symmetric positive definite problem for the dual variables, i.e., the Lagrange multipliers. This dual problem is solved by the conjugate gradient method with a Dirichlet preconditioner. In each iteration step, both subdomain problems and a coarse level problem are solved by a direct method. It is proved that the condition number of this preconditioned dual problem is independent of the number of subdomains and bounded from above by the square of the product of the inverse of the inf-sup constant of the discrete problem and the logarithm of the number of unknowns in the individual subdomains. Numerical experiments demonstrate the scalability of this new method.

1 Introduction

The finite element tearing and interconnecting(FETI) methods were first proposed by Farhat and Roux [10] for elliptic partial differential equations. In this method, the spatial domain is decomposed into nonoverlapping subdomains, and the interior subdomain variables are eliminated to form a Schur problem for the interface variables. Lagrange multipliers are then introduced to enforce continuity across the interface, and a symmetric positive semi-definite linear system for the Lagrange multipliers is solved by using the preconditioned conjugate gradient (PCG) method. This method has been shown to be numerically scalable for second order elliptic problems if a Dirichlet preconditioner is used. Thus, Mandel and Tezaur [21] have proved that the condition number grows at most as $C(1+\log(H/h))^3$ both in two and three dimensions, where H is the subdomain diameter and h is the element size, cf. also Tezaur [33]. Klawonn and Widlund [19] proposed new preconditioners of this type and proved that the condition numbers are bounded from above by $C(1+\log(H/h))^2$, a bound which is independent of possible jumps of the coefficients of the elliptic problem. The same condition number bound was also proved by Brenner [4] in the standard additive Schwarz framework. All these condition number bounds are independent of the number of subdomains and therefore imply the parallel scalability of the FETI methods.

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For fourth-order problems, a two-level FETI method was developed by Farhat and Mandel [9]. The main idea in this variant is that an extra set of Lagrange multipliers should be used to enforce the continuity at the subdomain corners in every step of the PCG algorithm. A condition number bound for the two-level FETI algorithm was given by Mandel, Tezaur, and Farhat [23]. A similar idea was used by Farhat, Lesoinne, and Pierson [11] to introduce the Dual-Primal FETI (FETI-DP) methods in which the continuity of the primal solution is enforced directly at the corners, i.e., the degrees of freedom at a cornerpoint are common to all subdomains sharing this corner. In [11], the FETI-DP methods were further refined to solve three-dimensional problems by introducing Lagrange multipliers to enforce a continuity constraint for the average of the solution on subdomain interface edges. This set of Lagrange multipliers, together with the corner variables, form the coarse problem of this FETI-DP method. This coarse, primal problem is necessary to obtain a satisfactory convergence rate for this method. A condition number bound of the dual-primal FETI methods was first given by Mandel and Tezaur [22] for two-dimensional problems. New preconditioners and condition number bounds for three-dimensional problems were further developed by Klawonn, Widlund, and Dryja [18].

In this paper, we develop a dual-primal FETI method for the incompressible Stokes problems in two dimensions and give a convergence analysis. The main difference here, is that we require the velocity component to have the same averages across each subdomain interface edge. This extra continuity constraint will stabilize the coarse level saddle point problem, and is also crucial for the derivation of a scalable condition number bound. The pressure space is decomposed into two orthogonal parts; the first part consists of subdomain interior pressures with zero average on each subdomain, and the second is spanned by the subdomain constant pressures with one average pressure for each subdomain. The velocity space is decomposed into three parts, the velocities interior to the subdomains, the velocities at subdomain corners and the velocities on the remaining part of the interface.

In our algorithm, we only consider finite element approximation with discontinuous pressure. This makes it possible to only enforce continuity of the velocity component across the subdomain interface. The continuity of the velocities at the subdomain corners is enforced directly in our algorithm, while the continuity of the velocities across the remaining interface is enforced by a set of Lagrange multipliers. By Schur complement procedures, the indefinite Stokes problem is reduced to a symmetric, positive definite problem for the dual variables, i.e., the Lagrange multipliers. The conjugate gradient method is used to solve this dual problem, with a Dirichlet preconditioner.

There has been extensive work on domain decomposition methods for saddle point problems, especially for incompressible Stokes problems; see [34, Chapter 9] and the references therein. Previous domain decomposition methods for incompressible Stokes equations have been based on primal iterative substructuring methods, cf. [1], [3], [5], [6], [7], [13], [24], [25], [26], [27], [28], [29], [32], on overlapping Schwarz methods, cf. [12], [14], [16], [30], and on block preconditioners, cf. [15], [17]. One advantage of the FETI type algorithms for incompressible Stokes problem is that the original indefinite problem is reduced to a symmetric positive definite problem of the interface Lagrange multiplier variables. Therefore not only is the number of degrees of freedom of the iterative solver largely reduced, but a preconditioned conjugate gradient method can also be used. The idea to develop a FETI type algorithm for incompressible Stokes problem is also inspired by the fact that the FETI methods are among the first domain decomposition methods that have demonstrated both numerical and parallel scalability for the solution of elliptic partial differential equations. Their ability to outperform several popular direct and iterative algorithms on both sequential and parallel computers has been extensively demonstrated, and has earned this family of methods a place in commercial finite element structural mechanics, cf. [8]. Our FETI-DP algorithm for Stokes problem inherits the advantages of the FETI methods for positive definite elliptic problems. We also note that some recent works of FETI algorithms for incompressible linear elasticity has already appeared, cf. [35].

The remainder of this paper is organized as follows. In Section 2, our domain decomposition procedure is described for solving the two-dimensional incompressible Stokes problem. In Section 3, the discrete Stokes harmonic extensions are introduced. Our FETI-DP algorithm for Stokes problem is derived in Section 4. In Sections 5 and 6, a condition number bound is given for the preconditioned FETI-DP algorithm, based on a general form of the FETI-DP algorithms. Numerical experiments to demonstrate the scalability of the new method are shown in Section 7.

Our dual-primal FETI algorithm has also been extended to solving stationary incompressible Navier-Stokes equations, cf. [20].

2 A Domain decomposition procedure for Stokes problems

We consider the following incompressible Stokes problem on a bounded, polyhedral domain Ω in two dimensions with a Dirichlet boundary condition,

$$\left\{ \begin{array}{rcll} -\Delta \mathbf{u} + \nabla p &=& \mathbf{f}, & \text{in } \Omega \ , \\ -\nabla \cdot \mathbf{u} &=& 0, & \text{in } \Omega \ , \\ \mathbf{u} &=& \mathbf{g}, & \text{on } \partial\Omega \ , \end{array} \right. \tag{1}$$

where the boundary data $\mathbf{g}$ satisfies the compatibility condition $\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n} = 0$. For simplicity, we assume that $\mathbf{g} = \mathbf{0}$ without losing any generality.

We solve this incompressible Stokes problem by using finite element methods. The domain Ω is triangulated into shape-regular elements of characteristic size h. We denote the finite element space for the velocity component by $\mathbf{W} \subset (H_0^1(\Omega))^2 = \{\mathbf{w} \in (H^1(\Omega))^2 \mid \mathbf{w} = \mathbf{0} \text{ on } \partial\Omega\}$, and that for the pressure component by $Q \subset L_0^2(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q = 0\}$. The discrete variational problem is: find $\mathbf{u} \in \mathbf{W}$ and $p \in Q$ such that

$$\left\{ \begin{array}{lcll} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &=& (\mathbf{f},\mathbf{v}), & \forall \mathbf{v} \in \mathbf{W}, \\ b(\mathbf{u},q) &=& 0, & \forall q \in Q \ , \end{array} \right. \tag{2}$$

where

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v} = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}, \quad b(\mathbf{u},q) = -\int_\Omega (\nabla\cdot\mathbf{u})q.$$

The domain Ω will be decomposed into subdomains, and we will use $a_i(\cdot,\cdot)$ and $b_i(\cdot,\cdot)$ to denote these bilinear forms restricted to a single subdomain Ω^i.

We assume inf-sup stability of the chosen mixed finite element space $\mathbf{W} \times Q$, i.e., that there exists a positive constant β, independent of h, such that

$$\sup_{\mathbf{w}\in\mathbf{W}} \frac{b(\mathbf{w},q)}{\|\mathbf{w}\|_{H^1}} \geq \beta \|q\|_{L^2}, \quad \forall q \in Q. \tag{3}$$

The matrix form of equation (2) is

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}. \tag{4}$$

The domain Ω is decomposed into N nonoverlapping polyhedral subdomains Ω^i, $i = 1, 2, ..., N$, of characteristic size H, with the finite element nodes on the boundaries of neighboring subdomains matching across the interface $\Gamma = (\cup \partial\Omega^i) \backslash \partial\Omega$. Γ is composed of subdomain edges, regarded as open sets, which are shared by two subdomains, and vertices which are end points of edges. We denote the edges of Ω^i by $\mathcal{E}^{ij}$, and the vertices of Ω^i by $\mathcal{V}^{ik}$.

We decompose the discrete velocity space $\mathbf{W}$ and the pressure space Q into

$$\begin{aligned} \mathbf{W} &= \mathbf{W}_I \oplus \mathbf{W}_\Gamma, \\ Q &= Q_I \oplus Q_0, \end{aligned} \tag{5}$$

where $\mathbf{W}_I$ and Q_I are the direct sums of subdomain interior velocity spaces $\mathbf{W}_I^i$, and subdomain interior pressure spaces Q_I^i, respectively, i.e.,

$$\mathbf{W}_I = \oplus_{i=1}^N \mathbf{W}_I^i, \quad Q_I = \oplus_{i=1}^N Q_I^i.$$

The subdomain interior velocity component $\mathbf{w}_I^i \in \mathbf{W}_I^i$ has its support in the subdomain Ω^i and equals zero on the subdomain interface $\Gamma \cap \partial\Omega^i$. The subdomain interior pressure component $q_I^i \in Q_I^i$ has zero average on the subdomain Ω^i and equals zero outside. $q_0 \in Q_0$ is the subdomain constant pressure part, which has a constant value q_0^i in subdomain Ω^i. These q_0^i satisfy

$$\sum_{i=1}^N q_0^i \; m(\Omega^i) = 0, \tag{6}$$

providing the global Stokes problem with a unique solution. Here $m(\Omega^i)$ is the measure of the subdomain Ω^i. $\mathbf{W}_\Gamma$ is the subdomain interface velocity space. All functions in $\mathbf{W}_\Gamma$ are continuous across the subdomain interface Γ. We use $\mathbf{W}_\Gamma^i$ to denote the component of $\mathbf{W}_\Gamma$ restricted to $\partial\Omega^i$, and use $\mathbf{W}^i$ to denote the velocity space on the subdomain Ω^i, i.e., $\mathbf{W}^i = \mathbf{W}_I^i \oplus \mathbf{W}_\Gamma^i$.

Using the decomposition (5) of the solution space, the linear system (4) can be written as

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Gamma I}^T & 0 \\ B_{II} & 0 & B_{I\Gamma} & 0 \\ A_{\Gamma I} & B_{I\Gamma}^T & A_{\Gamma\Gamma} & B_{0\Gamma}^T \\ 0 & 0 & B_{0\Gamma} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Gamma \\ p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Gamma \\ 0 \end{pmatrix}. \tag{7}$$

We note that, for the sake of simplicity, we make no distinction between the symbols of a vector and the corresponding finite element function, and also no distinction between a matrix and the corresponding finite element function operator. For example, $\mathbf{u}_\Gamma$ can denote either a vector in the vector space $\mathbf{W}_\Gamma$, or a finite element function in the finite element function space $\mathbf{W}_\Gamma$.

3 Discrete Stokes harmonic extensions

From equation (7), we define a Schur complement operator S_Γ by

$$S_\Gamma = A_{\Gamma\Gamma} - (A_{\Gamma I} \; B_{I\Gamma}^T) \begin{pmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Gamma I}^T \\ B_{I\Gamma} \end{pmatrix}, \tag{8}$$

which can be computed by subassembling subdomain Schur complement operators S_Γ^i, defined for the subdomain Ω^i by

$$S_\Gamma^i = A_{\Gamma\Gamma}^i - (A_{\Gamma I}^i \; B_{I\Gamma}^{i^T}) \begin{pmatrix} A_{II}^i & B_{II}^{i^T} \\ B_{II}^i & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Gamma I}^{i^T} \\ B_{I\Gamma}^i \end{pmatrix}. \tag{9}$$

We know, from the definition (9), that the action of S_Γ^i can be evaluated by solving a Dirichlet problem on the subdomain Ω^i. The following lemma shows that the subdomain Schur complements S_Γ^i are symmetric, positive semi-definite; this can be proved in the same way as Lemma 4 in Section 5.

Lemma 1 *The subdomain Schur complements S_Γ^i defined in (9) are symmetric, positive semi-definite, and they are singular for any subdomain with a boundary that does not intersect $\partial\Omega$.*

Each subdomain Schur complement operator S_Γ^i corresponds to the standard discrete Stokes harmonic extension operator $\mathcal{SH}^i$: $\mathbf{W}_\Gamma^i \to \mathbf{W}^i$, defined by: given a velocity $\mathbf{w}_\Gamma^i \in \mathbf{W}_\Gamma^i$, find $\mathcal{SH}^i\mathbf{w}_\Gamma^i \in \mathbf{W}^i$ and $p_I^i \in Q_I^i$ such that

$$\left\{ \begin{array}{lcll} a_i(\mathcal{SH}^i\mathbf{w}_\Gamma^i, \mathbf{v}^i) + b_i(\mathbf{v}^i, p_I^i) & = & 0, & \forall \mathbf{v}^i \in \mathbf{W}_I^i, \\ b_i(\mathcal{SH}^i\mathbf{w}_\Gamma^i, q_I^i) & = & 0, & \forall q_I^i \in Q_I^i, \\ \mathcal{SH}^i\mathbf{w}_\Gamma^i & = & \mathbf{w}_\Gamma^i, & \text{on } \partial\Omega^i. \end{array} \right. \tag{10}$$

The S_Γ^i-seminorm is defined on the finite element function space $\mathbf{W}_\Gamma^i$ by

$$|\mathbf{w}_\Gamma^i|_{S_\Gamma^i}^2 = \mathbf{w}_\Gamma^{i^T} S_\Gamma^i \mathbf{w}_\Gamma^i = a_i(\mathcal{SH}^i\mathbf{w}_\Gamma^i, \mathcal{SH}^i\mathbf{w}_\Gamma^i).$$

A global discrete Stokes extension operator $\mathcal{SH}$: $\mathbf{W}_\Gamma \to \mathbf{W}$, is defined by $\mathcal{SH}^i$, its restriction to the subspace $\mathbf{W}_\Gamma^i$. The S_Γ-seminorm is defined on the finite element function space $\mathbf{W}_\Gamma$, by

$$|\mathbf{w}_\Gamma|_{S_\Gamma}^2 = \sum_{i=1}^N |\mathbf{w}_\Gamma^i|_{S_\Gamma^i}^2 = \sum_{i=1}^N \mathbf{w}_\Gamma^{i^T} S_\Gamma^i \mathbf{w}_\Gamma^i = \mathbf{w}_\Gamma^T S_\Gamma \mathbf{w}_\Gamma = a(\mathcal{SH}\mathbf{w}_\Gamma, \mathcal{SH}\mathbf{w}_\Gamma).$$

The discrete Stokes extension operators $\mathcal{SH}^i$ and $\mathcal{SH}$ are the counterparts of the discrete harmonic extension operators $\mathcal{H}^i$ and $\mathcal{H}$, respectively. The subdomain discrete harmonic extension operator $\mathcal{H}^i : \mathbf{W}_\Gamma^i \to \mathbf{W}^i$, is defined by: given $\mathbf{w}_\Gamma^i \in \mathbf{W}_\Gamma^i$, find $\mathcal{H}^i\mathbf{w}_\Gamma^i \in \mathbf{W}^i$ such that

$$\left\{ \begin{array}{lcll} a_i(\mathcal{H}^i\mathbf{w}_\Gamma^i, \mathbf{v}^i) & = & 0, & \forall \mathbf{v}^i \in \mathbf{W}_I^i, \\ \mathcal{H}^i\mathbf{w}_\Gamma^i & = & \mathbf{w}_\Gamma^i, & \text{on } \partial\Omega^i. \end{array} \right. \tag{11}$$

The global discrete harmonic extension operator $\mathcal{H}$: $\mathbf{W}_\Gamma \to \mathbf{W}$, is defined by $\mathcal{H}^i$, its restriction on the subspace $\mathbf{W}_\Gamma^i$.

The following lemma can be found in Bjørstad and Widlund [2] for the case of piecewise linear elements and two dimensions. The tools necessary to extend this result to more general finite elements are provided in Widlund [36].

Lemma 2 *There exist positive constants C_1 and C_2, independent of H and h, such that*

$$C_1|\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)}^2 \le a(\mathcal{H}\mathbf{w}_\Gamma, \mathcal{H}\mathbf{w}_\Gamma) \le C_2|\mathbf{w}_\Gamma|_{H^{1/2}(\Gamma)}^2, \quad \forall \mathbf{w}_\Gamma \in \mathbf{W}_\Gamma,$$
$$C_1|\mathbf{w}_\Gamma^i|_{H^{1/2}(\partial\Omega^i)}^2 \le a_i(\mathcal{H}^i\mathbf{w}_\Gamma^i, \mathcal{H}^i\mathbf{w}_\Gamma^i) \le C_2|\mathbf{w}_\Gamma^i|_{H^{1/2}(\partial\Omega^i)}^2, \quad \forall \mathbf{w}_\Gamma^i \in \mathbf{W}_\Gamma^i.$$

The following result can be found in Bramble and Pasciak [3, Theorem 4.1], or in Pavarino and Widlund [27, Lemma 3.1],

Lemma 3 *There exist positive constants C_1 and C_2, independent of H and h, such that*

$$C_1\beta^2 a(\mathcal{SH}\mathbf{w}_\Gamma, \mathcal{SH}\mathbf{w}_\Gamma) \le a(\mathcal{H}\mathbf{w}_\Gamma, \mathcal{H}\mathbf{w}_\Gamma) \le C_2 a(\mathcal{SH}\mathbf{w}_\Gamma, \mathcal{SH}\mathbf{w}_\Gamma), \quad \forall \mathbf{w}_\Gamma \in \mathbf{W}_\Gamma,$$
$$C_1\beta^2 a_i(\mathcal{SH}^i\mathbf{w}_\Gamma^i, \mathcal{SH}^i\mathbf{w}_\Gamma^i) \le a_i(\mathcal{H}^i\mathbf{w}_\Gamma^i, \mathcal{H}^i\mathbf{w}_\Gamma^i) \le C_2 a_i(\mathcal{SH}^i\mathbf{w}_\Gamma^i, \mathcal{SH}^i\mathbf{w}_\Gamma^i), \quad \forall \mathbf{w}_\Gamma^i \in \mathbf{W}_\Gamma^i,$$

where β is the inf-sup stability constant defined in (3).

4 The augmented FETI-DP algorithm

The interface velocity space $\mathbf{W}_\Gamma$ is decomposed into a subdomain corner velocity subspace $\mathbf{W}_C$ and the remaining interface velocity subspace $\mathbf{W}_\Delta$, subject to a continuity constraint across Γ, i.e.,

$$\mathbf{W}_\Gamma = \{\mathbf{w} \in \mathbf{W}_C \oplus \mathbf{W}_\Delta \mid \mathbf{w} \text{ continuous across } \Gamma\} \ . \tag{12}$$

The continuity of the functions in $\mathbf{W}_C$ is enforced directly, i.e., the degrees of freedom at a cornerpoint are common to all subdomains sharing this corner. $\mathbf{W}_\Delta$ is the direct sum of subdomain interface velocity space, i.e.,

$$\mathbf{W}_\Delta = \oplus_{i=1}^N \mathbf{W}_\Delta^i.$$

The functions $\mathbf{w}_\Delta$ in the space $\mathbf{W}_\Delta$ are not necessarily continuous across Γ, and the continuity is enforced by

$$B_\Delta \mathbf{w}_\Delta = 0, \tag{13}$$

where the matrix B_Δ is a boolean matrix constructed from {0,1,-1} such that the values of $\mathbf{w}_\Delta$ coincide across the subdomain interface Γ when $B_\Delta \mathbf{w}_\Delta = 0$. Here we describe B_Δ as a boolean matrix to be consistent with most other literatures about FETI methods. Let us note that we always use a single subscript for these boolean type matrices, and use double subscripts for the matrices related in the formulation of saddle point problems, for example $B_{0\Gamma}$, $B_{I\Gamma}$, B_{II}, etc., in equation (7).

A set of redundant continuity constraints are introduced in the form of

$$Q_\Delta^T B_\Delta \mathbf{w}_\Delta = 0, \tag{14}$$

for any function $\mathbf{w}_\Delta \in \mathbf{W}_\Delta$, where the matrix Q_Δ is constructed such that, $Q_\Delta^T B_\Delta \mathbf{w}_\Delta = 0$ implies

$$\int_{\mathcal{E}^{ij}} (\mathbf{w}_\Delta^i - \mathbf{w}_\Delta^j) = 0, \quad \forall \mathcal{E}^{ij}. \tag{15}$$

By using the decomposition (12) of the space $\mathbf{W}_\Gamma$ and introducing Lagrange multipliers λ and μ to enforce the continuity constraints (13) and (14) for the functions in $\mathbf{W}_\Delta$, equation (7) can be written as

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Delta I}^T & A_{CI}^T & 0 & 0 & 0 \\ B_{II} & 0 & B_{I\Delta} & B_{IC} & 0 & 0 & 0 \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{C\Delta}^T & B_{0\Delta}^T & B_\Delta^T Q_\Delta & B_\Delta^T \\ A_{CI} & B_{IC}^T & A_{C\Delta} & A_{CC} & B_{0C}^T & 0 & 0 \\ 0 & 0 & B_{0\Delta} & B_{0C} & 0 & 0 & \\ 0 & 0 & Q_\Delta^T B_\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & B_\Delta & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_C \\ p_0 \\ \mu \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_C \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{16}$$

Thus, two sets of Lagrange multipliers λ and μ are introduced to enforce the continuity constraints given by equations (13) and (14). In fact, μ is redundant because $B_\Delta \mathbf{u}_\Delta = 0$ implies $Q_\Delta^T B_\Delta \mathbf{u}_\Delta = 0$. But in our algorithm, λ and μ are treated differently. We iterate on the dual variable λ, and the continuity condition $B_\Delta \mathbf{u}_\Delta = 0$ is not satisfied until convergence. However, $Q_\Delta^T B_\Delta \mathbf{u}_\Delta = 0$ will be satisfied throughout the iteration, because in each iteration step we solve a coarse level problem, which includes the Lagrange multipliers μ. We will see later that μ is crucial for the stability of the coarse level saddle point problem and for a scalable condition number bound of the preconditioned problem.

By using the notations

$$\tilde{\mathbf{u}}_r = \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \end{pmatrix}, \; \tilde{\mathbf{u}}_c = \begin{pmatrix} \mathbf{u}_C \\ p_0 \\ \mu \end{pmatrix}, \tag{17}$$

equation (16) can be written as,

$$\begin{pmatrix} K_{rr} & K_{rc} & B_r^T \\ K_{rc}^T & K_{cc} & 0 \\ B_r & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}}_r \\ \tilde{\mathbf{u}}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}}_r \\ \tilde{\mathbf{f}}_c \\ 0 \end{pmatrix}, \tag{18}$$

where $K_{rr}, K_{rc}, K_{cc}, B_r, \tilde{\mathbf{f}}_r$, and $\tilde{\mathbf{f}}_c$, are the corresponding block matrices and block vectors. K_{rr} is a block diagonal matrix, where each block corresponds to a subdomain saddle point problem, with corner velocity values given. These corner nodes remove the singularities of these subdomain problems, and therefore K_{rr} is nonsingular.

Our algorithm results from two consecutive elimination procedures applied to equation (18). We first eliminate the subdomain independent variables $\tilde{\mathbf{u}}_r$ and obtain

$$\begin{pmatrix} \tilde{K}_{cc} & \tilde{K}_{cl} \\ \tilde{K}_{cl}^T & \tilde{K}_{ll} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}}_c^* \\ d_l \end{pmatrix}, \tag{19}$$

where

$$\tilde{K}_{cc} = K_{cc} - K_{rc}^T K_{rr}^{-1} K_{rc}, \quad \tilde{K}_{ll} = -B_r K_{rr}^{-1} B_r^T, \quad \tilde{K}_{cl} = -K_{rc}^T K_{rr}^{-1} B_r^T,$$

and

$$\tilde{\mathbf{f}}_c^* = \tilde{\mathbf{f}}_c - K_{rc}^T K_{rr}^{-1} \tilde{\mathbf{f}}_r, \quad d_l = -B_r K_{rr}^{-1} \tilde{\mathbf{f}}_r .$$

We then eliminate $\tilde{\mathbf{u}}_c$ from equation (19), and obtain a linear system for the Lagrange multipliers λ,

$$(\tilde{K}_{ll} - \tilde{K}_{cl}^T \tilde{K}_{cc}^{-1} \tilde{K}_{cl})\lambda = d_l - \tilde{K}_{cl}^T \tilde{K}_{cc}^{-1} \tilde{\mathbf{f}}_c^* . \tag{20}$$

This is the dual problem of our nonpreconditioned FETI-DP algorithm for incompressible Stokes equations. Each iteration step requires solving independent subdomain problems, to multiply $\tilde{K}_{cl}$, $\tilde{K}_{cl}^T$, or $\tilde{K}_{ll}$ by a vector. $\tilde{K}_{cc}^{-1}$ is a coarse level solver; the non-singularity of the matrix $\tilde{K}_{cc}$ will be addressed in the next section in a general form. We will also show in Section 5 that the operator of this dual problem (20) is symmetric, positive definite, and that therefore a preconditioned conjugate gradient method can be used. After obtaining λ, we backsolve equations (19) and (18) to obtain $\tilde{\mathbf{u}}_c$ and $\tilde{\mathbf{u}}_r$, respectively. We notice that the coarse level problem matrix and subdomain problem matrices only need to be factored once.

The preconditioner used in our FETI-DP algorithm is the standard Dirichlet preconditioner, given by

$$B_r \left(A_{\Delta\Delta} - (A_{\Delta I} \; B_{I\Delta}^T) \begin{pmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Delta I}^T \\ B_{I\Delta} \end{pmatrix} \right) B_r^T, \tag{21}$$

which involves solving subdomain incompressible Stokes problems with Dirichlet boundary conditions.

Our FETI-DP algorithm can be called *augmented* due to the inclusion of μ among the coarse level problem variables $\tilde{\mathbf{u}}_c$ in equation (18), i.e., we augment the subdomain corners velocity part $\mathbf{u}_C$ with a set of Lagrange multiplier variables corresponding to all the subdomain edges. This augmentation enriches the velocity space of the coarse level saddle point problem and makes it stable. In Section 7, we also conduct numerical experiments with *non-augmented* FETI-DP algorithm, which can be derived in the same way as above, except that the constraint (15) is no longer enforced. In that case, μ no longer appears in equation (16), etc..

5 Dual-primal FETI algorithms in a general form

By solving a coarse level problem exactly in each iteration step in our augmented FETI-DP algorithm, the dual velocity part $\mathbf{u}_\Delta$, in equation (16), always satisfies the extra continuity constraint (15) throughout the iteration. Instead of introducing Lagrange multiplier μ to enforce (15), an equivalent approach is to include the edge cutoff functions $\theta_{\mathcal{E}^{ij}}$, as well as the vertex cutoff functions $\theta_{\mathcal{V}^{ik}}$, in the coarse level velocity space. Here $\theta_{\mathcal{E}^{ij}}$ is the discrete Stokes harmonic function which equals 1 at each grid point on the edge $\mathcal{E}^{ij}$ and vanishes at the other interface points; $\theta_{\mathcal{V}^{ik}}$ is the discrete Stokes harmonic extension of the standard nodal basis function associated with each vertex $\mathcal{V}^{ik}$. The interface velocity space $\mathbf{W}_\Gamma$ can then be written in the form of

$$\mathbf{W}_\Gamma = \left\{ \mathbf{w} \in \mathbf{W}_\Pi \oplus \widetilde{\mathbf{W}}_\Delta \mid \mathbf{w} \text{ continuous across } \Gamma \right\} . \tag{22}$$

The coarse level velocity space $\mathbf{W}_\Pi$ is spanned by the $\theta_{\mathcal{V}^{ik}}$ and $\theta_{\mathcal{E}^{ij}}$, associated with all the vertices and edges of the interface Γ. $\widetilde{\mathbf{W}}_\Delta$ is the dual part of the velocity space, and it is the direct sum of local subspaces $\widetilde{\mathbf{W}}^i_\Delta$, which are defined by

$$\widetilde{\mathbf{W}}^i_\Delta = \{ \mathbf{w}^i_\Delta \in \mathbf{W}^i_\Gamma \mid \mathbf{w}^i_\Delta(\mathcal{V}^{ik}) = 0, \overline{\mathbf{w}}^i_{\Delta,\mathcal{E}^{ij}} = 0, \ \forall \mathcal{V}^{ik}, \mathcal{E}^{ij} \subset \partial\Omega^i \},$$

with $\overline{\mathbf{w}}^i_{\Delta,\mathcal{E}^{ij}}$ defined by

$$\overline{\mathbf{w}}^i_{\Delta,\mathcal{E}^{ij}} = \frac{\int_{\mathcal{E}^{ij}} \mathbf{w}^i_\Delta d\mathbf{x}}{\int_{\mathcal{E}^{ij}} d\mathbf{x}}.$$

The functions in the primal subspace $\mathbf{W}_\Pi$ are continuous across Γ, since their degrees of freedom on Γ are shared by neighboring subdomains. The functions in the dual subspace $\widetilde{\mathbf{W}}_\Delta$ are direct sum of subdomain components, and are not necessarily continuous across Γ. With the decompositions of the velocity and pressure spaces as in equations (5) and (22), we have the following problem: find a function $(\mathbf{u}_I, p_I, \mathbf{u}_\Pi, \mathbf{u}_\Delta, p_0) \in (\mathbf{W}_I, Q_I, \mathbf{W}_\Pi, \widetilde{\mathbf{W}}_\Delta, Q_0)$ such that

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & A_{\Delta I}^T & 0 \\ B_{II} & 0 & B_{I\Pi} & B_{I\Delta} & 0 \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & A_{\Delta\Pi}^T & B_{0\Pi}^T \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & A_{\Delta\Delta} & B_{0\Delta}^T \\ 0 & 0 & B_{0\Pi} & B_{0\Delta} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Pi \\ \mathbf{u}_\Delta \\ p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Pi \\ \mathbf{f}_\Delta \\ 0 \end{pmatrix}, \tag{23}$$

where the dual velocity part $\mathbf{u}_\Delta$ is required to be continuous across the subdomain interface Γ.

By defining a Schur complement operator $\tilde{S}$ as

$$\begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T \\ B_{II} & 0 & B_{I\Pi} & 0 & B_{I\Delta} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & B_{0\Pi}^T & A_{\Delta\Pi}^T \\ 0 & 0 & B_{0\Pi} & 0 & B_{0\Delta} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & B_{0\Delta}^T & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{w}_I \\ p_I \\ \mathbf{w}_\Pi \\ p_0 \\ \mathbf{w}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{S}\mathbf{w}_\Delta \end{pmatrix}, \tag{24}$$

solving equation (23) is reduced to solving the following interface problem

$$\tilde{S}\mathbf{u}_\Delta = \mathbf{f}_\Delta^*. \tag{25}$$

We introduce Lagrange multipliers λ to enforce the continuity of $\mathbf{u}_\Delta$ across Γ and obtain the following saddle point problem,

$$\begin{pmatrix} \tilde{S} & B_\Delta^T \\ B_\Delta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Delta^* \\ 0 \end{pmatrix}, \tag{26}$$

where the matrix B_Δ is defined as in equation (13). Note that we are not requiring the pressure to be continuous across the subdomain interface in our algorithm, since we only consider finite elements with a discontinuous pressure space.

By using an additional Schur complement procedure, equation (26) is further reduced to solving the following problem: find the Lagrange multipliers $\lambda \in \Lambda = B_\Delta \widetilde{\mathbf{W}}_\Delta$, such that:

$$B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta \tilde{S}^{-1} \mathbf{f}_\Delta^*. \tag{27}$$

In the remainder of this section, we will show that $\tilde{S}$ is well defined on the space $\widetilde{\mathbf{W}}_\Delta$, and that it is symmetric positive definite. Given a vector $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, to determine $\tilde{S}\mathbf{w}_\Delta$ from equation (24), we first solve the following coarse level saddle point problem

$$\begin{pmatrix} \widehat{A}_{\Pi\Pi} & B_{0\Pi}^T \\ B_{0\Pi} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{w}_\Pi \\ p_0 \end{pmatrix} = \begin{pmatrix} \mathbf{g}_\Pi \\ 0 \end{pmatrix}, \tag{28}$$

where

$$\widehat{A}_{\Pi\Pi} = A_{\Pi\Pi} - \begin{pmatrix} A_{\Pi I} & B_{I\Pi}^T \end{pmatrix} \begin{pmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^T \\ B_{I\Pi} \end{pmatrix},$$

and

$$\begin{pmatrix} \mathbf{g}_\Pi \\ 0 \end{pmatrix} = -\begin{pmatrix} A_{\Delta\Pi}^T \\ B_{0\Delta} \end{pmatrix} \mathbf{w}_\Delta + \begin{pmatrix} A_{\Pi I} & B_{I\Pi}^T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{\Delta I}^T \\ B_{I\Delta} \end{pmatrix} \mathbf{w}_\Delta,$$

where we know $B_{0\Delta}\mathbf{w}_\Delta = 0$, since $\mathbf{w}_\Delta \in \widetilde{W}_\Delta$. With the coarse level velocity space $\mathbf{W}_\Pi$ spanned by both vertex and edge cutoff functions $\theta_{\mathcal{V}^{ik}}$ and $\theta_{\mathcal{E}^{ij}}$, the coarse level problem (28) has a unique solution since $\widehat{A}_{\Pi\Pi}$ is positive definite and ${B_{0\Pi}}^T$ has full column rank. Also this coarse level saddle point problem is more stable than that in the case of non-augmented algorithms, where $\mathbf{W}_\Pi$ is spanned only by vertex cutoff functions $\theta_{\mathcal{V}^{ik}}$. Figure 1 gives an illustration of the mixed finite element space of the coarse level saddle point problem, for a 3 by 3 subdomains decomposition. In these figures, we use a circle to represent a pressure degree of freedom, and a cross for a velocity degree of freedom. The left figure corresponds to the case where the continuity constraint (15) is enforced and the right one corresponds to the case where it is not.

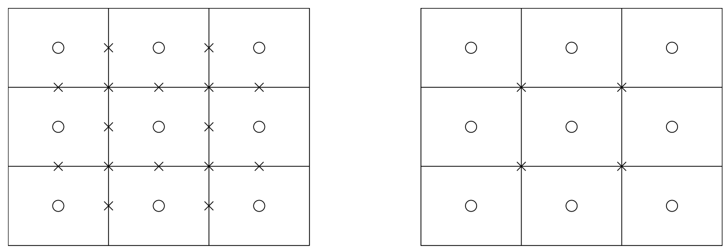

Figure 1: Mixed finite element space of the coarse level saddle point problem

The equivalent bilinear form of the coarse level problem (28) will be given in the proof of Lemma 7.

After solving the coarse level problem (28), we need to solve subdomain Dirichlet problems with boundary data given by $\mathbf{w}_\Delta$ and $\mathbf{w}_\Pi$ to obtain $\mathbf{w}_I$ and p_I in equation (24), and finally obtain $\tilde{S}\mathbf{w}_\Delta$.

Here we need to mention that the coarse level problem (28) differs from the coarse level problem solved in each iteration step. From equation (27), we see that, in each iteration step, we only multiply $\tilde{S}^{-1}$, not $\tilde{S}$, by a vector. To implement $\tilde{S}^{-1}$, the coarse level problem matrix is, by equation (24),

$$\begin{pmatrix} A_{\Pi\Pi} & B_{0\Pi}^T \\ B_{0\Pi} & 0 \end{pmatrix} - \begin{pmatrix} A_{\Pi I} & B_{I\Pi}^T & A_{\Delta\Pi}^T \\ 0 & 0 & B_{0\Delta} \end{pmatrix} \begin{pmatrix} A_{II} & B_{II}^T & A_{\Delta I}^T \\ B_{II} & 0 & B_{I\Delta} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^T & 0 \\ B_{I\Pi} & 0 \\ A_{\Delta\Pi} & B_{0\Delta}^T \end{pmatrix}, \tag{29}$$

which is equivalent to the $\tilde{K}_{cc}$ matrix in equation (20). If we write the matrix (29) in the form of

$$\begin{pmatrix} D_{\Pi\Pi} & E_{0\Pi}^T \\ E_{0\Pi} & F_{00} \end{pmatrix},$$

then we can see that $D_{\Pi\Pi}$ is symmetric positive definite, and F_{00} symmetric negative definite. Therefore the whole matrix (29) is a non-singular matrix.

Lemma 4 *$\tilde{S}$ is symmetric, positive definite on the space $\widetilde{\mathbf{W}}_\Delta$.*

Proof: It is easy to see, from its definition in (24), that $\tilde{S}$ is symmetric. We next just need to show that $\mathbf{w}_\Delta^T \tilde{S}\mathbf{w}_\Delta > 0$, for any nonzero function $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$. For any given function $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we can always find a vector $(\mathbf{w}_I, p_I, \mathbf{w}_\Pi, p_0)$ such that equation (24) is satisfied. Therefore,

$$\begin{aligned} & \mathbf{w}_\Delta^T \tilde{S}\mathbf{w}_\Delta \\ = & \begin{pmatrix} \mathbf{w}_I \\ p_I \\ \mathbf{w}_\Pi \\ p_0 \\ \mathbf{w}_\Delta \end{pmatrix}^T \begin{pmatrix} A_{II} & B_{II}^T & A_{\Pi I}^T & 0 & A_{\Delta I}^T \\ B_{II} & 0 & B_{I\Pi} & 0 & B_{I\Delta} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Pi} & B_{0\Pi}^T & A_{\Delta\Pi}^T \\ 0 & 0 & B_{0\Pi} & 0 & B_{0\Delta} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Pi} & B_{0\Delta}^T & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{w}_I \\ p_I \\ \mathbf{w}_\Pi \\ p_0 \\ \mathbf{w}_\Delta \end{pmatrix} \\ = & \begin{pmatrix} \mathbf{w}_I \\ \mathbf{w}_\Pi \\ \mathbf{w}_\Delta \end{pmatrix}^T \begin{pmatrix} A_{II} & A_{\Pi I}^T & A_{\Delta I}^T \\ A_{\Pi I} & A_{\Pi\Pi} & A_{\Delta\Pi}^T \\ A_{\Delta I} & A_{\Delta\Pi} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{w}_I \\ \mathbf{w}_\Pi \\ \mathbf{w}_\Delta \end{pmatrix} \end{aligned}$$

$$
+2\begin{pmatrix} p_I \\ p_0 \end{pmatrix}^T \begin{pmatrix} B_{II} & B_{I\Pi} & B_{I\Delta} \\ 0 & B_{0\Pi} & B_{0\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{w}_I \\ \mathbf{w}_\Pi \\ \mathbf{w}_\Delta \end{pmatrix} + \begin{pmatrix} p_I \\ p_0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_I \\ p_0 \end{pmatrix}
$$

$$
= \begin{pmatrix} \mathbf{w}_I \\ \mathbf{w}_\Pi \\ \mathbf{w}_\Delta \end{pmatrix}^T \begin{pmatrix} A_{II} & A_{\Pi I}^T & A_{\Delta I}^T \\ A_{\Pi I} & A_{\Pi\Pi} & A_{\Delta\Pi}^T \\ A_{\Delta I} & A_{\Delta\Pi} & A_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \mathbf{w}_I \\ \mathbf{w}_\Pi \\ \mathbf{w}_\Delta \end{pmatrix},
$$

where the last equality results from $B_{II}\mathbf{w}_I + B_{I\Pi}\mathbf{w}_\Pi + B_{I\Delta}\mathbf{w}_\Delta = 0$ and $B_{0\Pi}\mathbf{w}_\Pi + B_{0\Delta}\mathbf{w}_\Delta = 0$, because the vector $(\mathbf{w}_I, p_I, \mathbf{w}_\Pi, p_0, \mathbf{w}_\Delta)$ satisfies equation (24). Since the velocity is continuous at the subdomain corners, we know that the matrix

$$
\begin{pmatrix} A_{II} & A_{\Pi I}^T & A_{\Delta I}^T \\ A_{\Pi I} & A_{\Pi\Pi} & A_{\Delta\Pi}^T \\ A_{\Delta I} & A_{\Delta\Pi} & A_{\Delta\Delta} \end{pmatrix}
$$

is a symmetric, positive definite discretization of a direct sum of Laplace operators with Dirichlet boundary conditions. Therefore we find that $\mathbf{w}_\Delta^T \tilde{S} \mathbf{w}_\Delta > 0$, for any nonzero function $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$.

□

Since $\tilde{S}$ is symmetric, positive definite on the space $\widetilde{\mathbf{W}}_\Delta$, we can define an $\tilde{S}$-norm on $\widetilde{\mathbf{W}}_\Delta$, i.e.,

$$
|\mathbf{w}_\Delta|_{\tilde{S}}^2 = \mathbf{w}_\Delta^T \tilde{S} \mathbf{w}_\Delta, \quad \forall \mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta.
$$

We know, from the proof of Lemma 4, that for any $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, $|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}_\Gamma|_{S_\Gamma}$, where $\mathbf{w}_\Gamma = \mathbf{w}_\Pi + \mathbf{w}_\Delta$, with $\mathbf{w}_\Pi$ determined by solving the coarse level saddle point problem (28).

6 Condition number bound

With the definition of S_Γ given in equation (8), the Dirichlet preconditioner (21) can be written as $B_\Delta S_\Gamma B_\Delta^T$, and the preconditioned problem is: find $\lambda \in \Lambda = B_\Delta \widetilde{\mathbf{W}}_\Delta$, such that:

$$
B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T \lambda = B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} \mathbf{f}_\Delta^* \ . \tag{30}
$$

When we use a Krylov subspace iterative method to solve equation (30), both S_Γ and $\tilde{S}^{-1}$ are always applied to vectors in the space $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$. In order to use a conjugate gradient method to solve the linear system (30), we have to show that both S_Γ and $\tilde{S}^{-1}$ are symmetric positive definite when restricted to the appropriate subspace. From Lemma 1, we know that S_Γ is symmetric positive semidefinite on the space $\mathbf{W}_\Gamma$ and it is singular because of the interior floating subdomains. Now on the space $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$, each interior floating subdomain becomes fixed, because the velocity equals zero at each subdomain vertex. Therefore S_Γ becomes nonsingular and is positive definite on the space $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$. We also know, from Lemma 4, that $\tilde{S}^{-1}$ is symmetric, positive definite on the space $\widetilde{\mathbf{W}}_\Delta$. The following lemma shows that $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$ is a subspace of $\widetilde{\mathbf{W}}_\Delta$, and therefore $\tilde{S}^{-1}$ is also symmetric, positive definite on $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta$.

Lemma 5 $B_\Delta^T B_\Delta \widetilde{\mathbf{W}}_\Delta \subset \widetilde{\mathbf{W}}_\Delta$.

Proof: Given $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we need to show that, on each subdomain interface edge $\mathcal{E}^{ij}$,

$$\int_{\mathcal{E}^{ij}} (B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i = 0, \quad \text{and} \quad \int_{\mathcal{E}^{ij}} (B_\Delta^T B_\Delta \mathbf{w}_\Delta)^j = 0.$$

This is easily verified by noticing that,

$$(B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i_{\mathcal{E}^{ij}} = (\mathbf{w}_\Delta^i - \mathbf{w}_\Delta^j)_{\mathcal{E}^{ij}}, \quad \text{and} \quad (B_\Delta^T B_\Delta \mathbf{w}_\Delta)^j_{\mathcal{E}^{ij}} = (\mathbf{w}_\Delta^j - \mathbf{w}_\Delta^i)_{\mathcal{E}^{ij}},$$

and that

$$\int_{\mathcal{E}^{ij}} \mathbf{w}_\Delta^i = \int_{\mathcal{E}^{ij}} \mathbf{w}_\Delta^j = 0,$$

because $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$.

□

In the remainder of this section, we give a condition number bound for the preconditioned operator $B_\Delta S_\Gamma B_\Delta^T B_\Delta \tilde{S}^{-1} B_\Delta^T$. We first give the following lemma, which can be found in [37, Lemma 3.3].

Lemma 6 *Let $\mathbf{w}_\Gamma \in \mathbf{W}_\Gamma$, and let $I^H \mathbf{w}_\Gamma$ be the linear interpolant using the values at the subdomain vertices. Then*

$$\sum_{\mathcal{E}^{ik} \in \partial\Omega^i} |\mathbf{w}_\Gamma - I^H \mathbf{w}_\Gamma|^2_{H_{00}^{1/2}(\mathcal{E}^{ik})} \leq C(1 + \log(H/h))^2 |\mathbf{w}_\Gamma|^2_{H^{1/2}(\partial\Omega^i)} .$$

Lemma 7 *For any $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, $|\mathbf{w}_\Delta|_{\tilde{S}} \leq 2|\mathbf{w}_\Delta|_{S_\Gamma}$.*

Proof: Given $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, in order to compute its $\tilde{S}$-norm, we need to determine the coarse level velocity component $\mathbf{w}_\Pi$ by solving equation (28). We know that $|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}_\Gamma|_{S_\Gamma}$, where $\mathbf{w}_\Gamma = \mathbf{w}_\Pi + \mathbf{w}_\Delta$. Therefore,

$$|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}_\Gamma|_{S_\Gamma} \leq |\mathbf{w}_\Delta|_{S_\Gamma} + |\mathbf{w}_\Pi|_{S_\Gamma}.$$

We now bound $|\mathbf{w}_\Pi|_{S_\Gamma}$ by $|\mathbf{w}_\Delta|_{S_\Gamma}$. The equivalent bilinear form of the coarse level saddle point problem (28) is: given $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, find $\mathbf{w}_\Pi \in \mathbf{W}_\Pi$ and $p_0 \in Q_0$ such that

$$\left\{ \begin{array}{lcll} a(\mathcal{SH}(\mathbf{w}_\Pi + \mathbf{w}_\Delta), \mathcal{SH}\mathbf{v}_\Pi) + b(\mathcal{SH}\mathbf{v}_\Pi, p_0) & = & 0, & \forall \mathbf{v}_\Pi \in \mathbf{W}_\Pi, \\ b(\mathcal{SH}(\mathbf{w}_\Pi + \mathbf{w}_\Delta), q_0) & = & 0, & \forall q_0 \in Q_0. \end{array} \right. \tag{31}$$

Since $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, we know that each of its subdomain components $\mathbf{w}_\Delta^i$ satisfies $\mathbf{w}_\Delta^i(\mathcal{V}^{ik}) = 0$ and $\int_{\mathcal{E}^{ij}} \mathbf{w}_\Delta^i d\mathbf{x} = 0$, for all $\mathcal{V}^{ik}$ and $\mathcal{E}^{ij}$. Therefore,

$$b(\mathcal{SH}\mathbf{w}_\Delta, q_0) = \sum_{i=1}^{N} b(\mathcal{SH}\mathbf{w}_\Delta^i, q_0^i) = 0,$$

and the problem (31) becomes

$$\left\{ \begin{array}{lll} a(\mathcal{SH}\mathbf{w}_\Pi, \mathcal{SH}\mathbf{v}_\Pi) + b(\mathcal{SH}\mathbf{v}_\Pi, p_0) & = -a(\mathcal{SH}\mathbf{w}_\Delta, \mathcal{SH}\mathbf{v}_\Pi), & \forall \mathbf{v}_\Pi \in \mathbf{W}_\Pi, \\ b(\mathcal{SH}\mathbf{w}_\Pi, q_0) & = 0, & \forall q_0 \in Q_0. \end{array} \right. \tag{32}$$

Taking $\mathbf{v}_\Pi = \mathbf{w}_\Pi$ in the first of equations (32), and using the second equation, we have

$$a(\mathcal{SH}\mathbf{w}_\Pi, \mathcal{SH}\mathbf{w}_\Pi) = -a(\mathcal{SH}\mathbf{w}_\Delta, \mathcal{SH}\mathbf{w}_\Pi),$$

which implies

$$|\mathbf{w}_\Pi|_{S_\Gamma} \le |\mathbf{w}_\Delta|_{S_\Gamma}.$$

Therefore,

$$|\mathbf{w}_\Delta|_{\tilde{S}} \le |\mathbf{w}_\Delta|_{S_\Gamma} + |\mathbf{w}_\Pi|_{S_\Gamma} \le 2|\mathbf{w}_\Delta|_{S_\Gamma}.$$

□

Remark: The inf-sup stability estimate of the coarse level saddle point problem (31) is not used in the proof of Lemma 7, since for each $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, $b(\mathcal{SH}\mathbf{w}_\Delta, q_0) = 0$. *Without the extra continuity constraint (15) enforced,* $b(\mathcal{SH}\mathbf{w}_\Delta, q_0)$ *is not necessarily zero, and to obtain the similar result as Lemma 7, at least a scalable inf-sup stability estimate of the coarse level problem (31) will be required. Also note that Lemma 7 is a result about the stability of the operator* $\tilde{S}$. *We have already seen in Section 5 that the extra continuity constraint (15) are important for the stability of the operator* $\tilde{S}$.

The proof of the following lemma is very similar to that of [22, Theorem 4.5], cf., also [18, Lemma 9].

Lemma 8 *For any* $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$, *we have,*

$$|B_\Delta^T B_\Delta \mathbf{w}_\Delta|_{S_\Gamma}^2 \le C\frac{1}{\beta^2}(1+\log(H/h))^2 |\mathbf{w}_\Delta|_{\tilde{S}}^2,$$

where β *is the inf-sup constant of the original problem defined in (3),* C *is a positive constant independent of* h *and* H.

Proof: We consider an arbitrary $\mathbf{w}_\Delta \in \widetilde{\mathbf{W}}_\Delta$. In order to compute its $\tilde{S}-$norm, we need to determine the element $\mathbf{w}_\Gamma = \mathbf{w}_\Pi + \mathbf{w}_\Delta \in \mathbf{W}_\Gamma$ with $\mathbf{w}_\Pi \in \mathbf{W}_\Pi$, which satisfies equation (24). Then, we know that $|\mathbf{w}_\Delta|_{\tilde{S}} = |\mathbf{w}_\Gamma|_{S_\Gamma}$. We next note that we can subtract any continuous function from $\mathbf{w}_\Delta$ without changing the values of $B_\Delta^T B_\Delta \mathbf{w}_\Delta$; thus, $B_\Delta^T B_\Delta \mathbf{w}_\Delta = B_\Delta^T B_\Delta(\mathbf{w}_\Delta + \mathbf{w}_\Pi - I^H \mathbf{w}_\Pi)$, where $I^H\mathbf{w}_\Pi$ is the linear interpolant on the subdomain boundary, from the values of $\mathbf{w}_\Pi$ at the subdomain vertices.

We have to estimate

$$|B_\Delta^T B_\Delta \mathbf{w}_\Delta|_{S_\Gamma}^2 = |B_\Delta^T B_\Delta(\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma)|_{S_\Gamma}^2.$$

We can therefore focus on the estimate of the contribution from a single subdomain. Noticing that $(B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i$ vanishes at the subdomain corners, we can split $(B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i$ as

$$(B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i = \sum_{\mathcal{E}^{ij} \subset \partial\Omega^i} (B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i_{\mathcal{E}^{ij}}.$$

On each edge $\mathcal{E}^{ij}$, we know that

$$(B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i_{\mathcal{E}^{ij}} = \left(B_\Delta^T B_\Delta(\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma)\right)^i_{\mathcal{E}^{ij}} = \left(\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma\right)^i_{\mathcal{E}^{ij}} - \left(\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma\right)^j_{\mathcal{E}^{ij}}.$$

We have to estimate its S_Γ^i-norm. We have, from Lemmas 2 and 3,

$$\begin{aligned} |(B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i_{\mathcal{E}^{ij}}|_{S_\Gamma^i}^2 &= |\left(\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma\right)^i_{\mathcal{E}^{ij}} - \left(\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma\right)^j_{\mathcal{E}^{ij}}|_{S_\Gamma^i}^2 \\ &\le \frac{C}{\beta^2} |\left(\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma\right)^i - \left(\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma\right)^j|_{H_{00}^{1/2}(\mathcal{E}^{ij})}^2 \\ &\le \frac{C}{\beta^2}\left(|\left(\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma\right)^i|_{H_{00}^{1/2}(\mathcal{E}^{ij})}^2 + |\left(\mathbf{w}_\Gamma - I^H\mathbf{w}_\Gamma\right)^j|_{H_{00}^{1/2}(\mathcal{E}^{ij})}^2 \right), \end{aligned}$$

where the two terms on the right can be bounded by $C\frac{1}{\beta^2}(1+\log(H/h))^2|\mathbf{w}_\Gamma^i|^2_{H^{1/2}(\partial\Omega^i)}$ and $C\frac{1}{\beta^2}(1+\log(H/h))^2|\mathbf{w}_\Gamma^k|^2_{H^{1/2}(\partial\Omega^j)}$, respectively, by using Lemma 6. Then, by using Lemma 3 again, we have

$$|(B_\Delta^T B_\Delta \mathbf{w}_\Delta)^i|^2_{S_\Gamma^i} \leq C\frac{1}{\beta^2}(1+\log(H/h))^2 \left(|\mathbf{w}_\Gamma^i|^2_{S_\Gamma^i} + \sum_{j\in\mathcal{N}^i} |\mathbf{w}_\Gamma^j|^2_{S_\Gamma^j} \right) ,$$

where $\mathcal{N}^i$ is the set of indices for subdomains which have an edge in common with subdomain Ω^i.

□

Using Lemmas 7 and 8, we can prove Theorem 1, in the same way as [22, Theorem 4.4] and [18, Theorem 1]. Lemma 7 is used to obtain the lower bound:

$$\lambda^T F \lambda \geq \frac{1}{2}\lambda^T M \lambda, \quad \forall \lambda \in \Lambda,$$

and Lemma 8 is used to obtain the upper bound:

$$\lambda^T F \lambda \leq C\frac{1}{\beta^2}(1+\log(H/h))^2 \lambda^T M \lambda, \quad \forall \lambda \in \Lambda,$$

with $F = B_\Delta \tilde{S}^{-1} B_\Delta^T$ and $M^{-1} = B_\Delta S_\Gamma B_\Delta^T$. We have therefore completed the proof of our main result.

Theorem 1 *The condition number of the preconditioned linear system (30) is bounded from above by $C\frac{1}{\beta^2}(1+\log(H/h))^2$, where β is the inf-sup constant of the original problem defined in (3), C is a positive constant independent of h and H.*

7 Numerical results

We have tested our augmented FETI-DP methods by solving several incompressible fluid problems. In each case, the numerical results are consistent with the condition number bound obtained in Theorem 1 for the preconditioned augmented FETI-DP algorithm. We first solve a lid-driven-cavity problem, where $\Omega = [0,1]\times[0,1]$ with boundary condition $\mathbf{g} = (1,0)$ on the upper side $y = 1$, and $\mathbf{g} = \mathbf{0}$ on the three other sides. Ω is triangularized by a uniform mesh, shown in Figure 2. The mixed finite elements that we are using, is also shown in Figure 2, where the velocity is linear on each triangle and the pressure is constant on the union of four triangles. The inf-sup stability of this mixed finite elements can be easily proved by using the macroelement technique given in [31].

The preconditioned linear system (30) was solved by using a conjugate gradient method, with an initial guess $\lambda = 0$. The iterations were stopped when the residual of the dual problem had been reduced by 10^{-6}. We compared the iteration counts of the augmented FETI-DP algorithm, where the extra continuity constraints (15) were satisfied throughout the iteration, with those of the non-augmented FETI-DP algorithm.

Figure 3 shows the CG iteration counts of the FETI-DP algorithms for different mesh sizes. From the left figure, we can see that the convergence rate of the augmented FETI-DP algorithm is independent of the number of subdomains when the size of subdomain problems is fixed; the convergence rate of the non-augmented FETI-DP algorithm become deteriorated

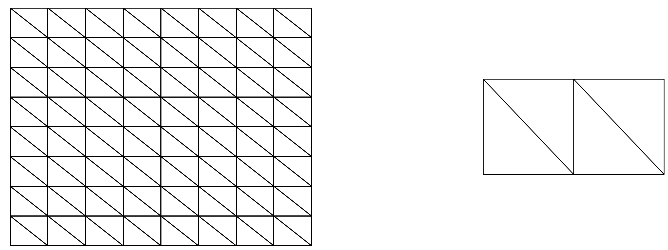

Figure 2: The mesh and the mixed finite elements

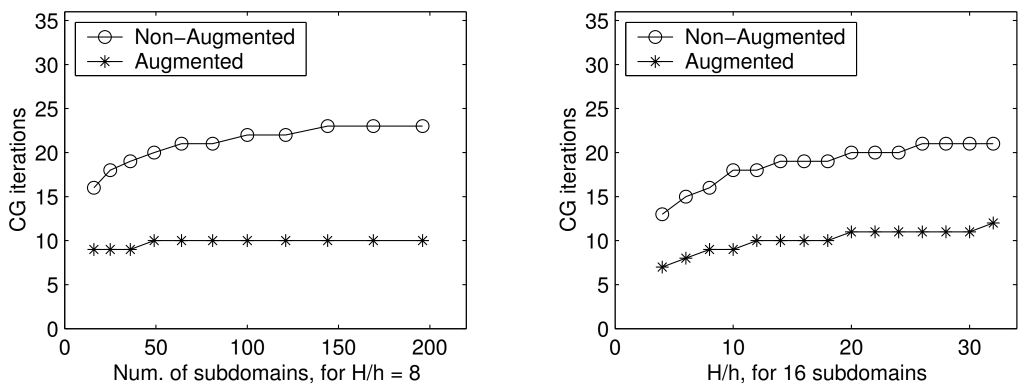

Figure 3: Scalability for solving lid-driver-cavity problem

with the increase of the number of subdomains. From the right figure in Figure 3, we can see that the convergence rate of both the augmented and non-augmented FETI-DP algorithms depends only slightly on the size of subdomain problems.

We have also implemented our algorithms to simulate a two-dimensional incompressible flow through an extrusion die, where the domain Ω and the boundary data are shown on the top in Figure 4. The boundary data $\mathbf{g}$ is chosen such that it satisfies the compatibility condition $\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n} = 0$ for incompressible problem. A typical decomposition of the domain Ω is also shown in Figure 4. The iteration counts of both the augmented and non-augmented FETI-DP algorithms are shown in Figure 5 for different mesh sizes. We can see that the convergence rate of the augmented algorithm is scalable and is as good as for solving the lid-driven-cavity problem. Without the edge continuity constraints being enforced, the convergence rate of the non-augmented FETI-DP algorithm depends on both the number of subdomains and the size of subdomain problems and is not scalable.

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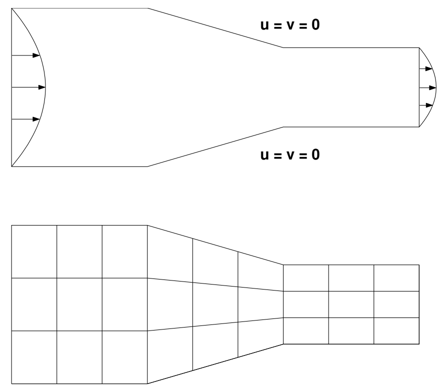

Figure 4: Upper: an extrusion die with Dirichlet boundary data; Lower: a typical decomposition of domain

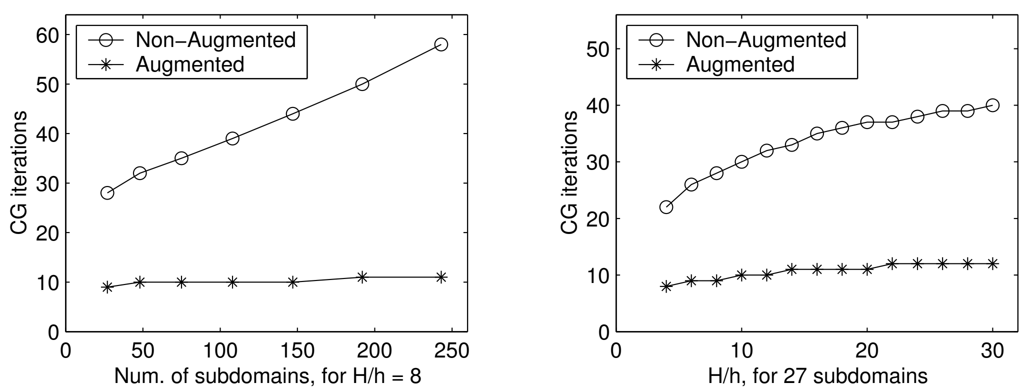

Figure 5: Scalability for solving extrusion die problem

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