CONVERGENCE ANALYSIS OF A BALANCING DOMAIN DECOMPOSITION METHOD FOR SOLVING A CLASS OF INDEFINITE LINEAR SYSTEMS

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Abstract. A variant of balancing domain decomposition method by constraints (BDDC) is proposed for solving a class of indefinite systems of linear equations of the form $(K - \sigma^2 M)u = f$, which arise from solving eigenvalue problems when an inverse shifted method is used and also from the finite element discretization of Helmholtz equations. Here both K and M are symmetric positive definite. The proposed BDDC method is closely related to the previous dual-primal finite element tearing and interconnecting method for solving this type of problems (Appl. Numer. Math., 54:150–166, 2005), where a coarse level problem containing certain free-space solutions of the inherent homogeneous PDE is used in the algorithm. Under the condition that the diameters of the subdomains are small enough, the convergence rate of the proposed algorithm is established which depends polylogarithmically on the dimension of the individual subdomain problems and which improves with a decrease of the subdomain diameters. These results are supported by numerical experiments of solving a two-dimensional problem.

Key words. domain decomposition, preconditioner, FETI, BDDC, indefinite, non-conforming

AMS subject classifications. 65F10, 65N30, 65N55

1. Introduction. Domain decomposition methods have been widely used and studied for solving large symmetric, positive definite linear systems arising from the finite element discretization of elliptic partial differential equations; theories on their convergence rates are well developed for such problems; see [48, 46, 43] and the references therein. Domain decomposition methods have also been applied to solving indefinite and nonsymmetric problems; cf. [1, 3, 4, 7, 8, 9, 23, 27, 28, 29, 35, 36, 41, 42, 47, 49]. A two-level overlapping Schwarz method was studied by Cai and Widlund [8] for solving indefinite elliptic problems, where they used a perturbation approach in the analysis to overcome the difficulty introduced by the indefiniteness of the problem and established that the convergence rate of the algorithm is independent of the mesh size if the coarse level mesh is fine enough. Such an approach was also used by Gopalakrishnan and Pasciak [23] and by Gopalakrishnan, Pasciak, and Demkowicz [24] in their analysis of overlapping Schwarz methods and multigrid methods for solving time harmonic Maxwell equations. For some other results using the perturbation approach in the analysis of domain decomposition methods for indefinite problems, see Xu [55] and Vassilevski [53].

The balancing domain decomposition methods by constraints (BDDC) were introduced by Dohrmann [13] for solving symmetric positive definite problems; see also Fragakis and Papadrakakis [22], and Cros [11]. They represent an interesting redesign of the Neumann-Neumann algorithms with the coarse, global component expressed in terms of a set of primal constraints. Spectral equivalence between the BDDC al-

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gorithms and the dual-primal finite element tearing and interconnecting algorithms (FETI-DP) has been proven by Mandel, Dohrmann, and Tezaur [39]; see also Li and Widlund [37], Brenner and Sung [6]. In these papers, it is established for the symmetric positive definite case that the preconditioned operators of a pair of BDDC and FETI-DP algorithms, with the same primal constraints, have the same eigenvalues except possibly those equal to 0 or 1.

In this paper, a BDDC method is proposed and analyzed for solving a class of indefinite system of linear equations

$$(K - \sigma^2 M)u = f, \tag{1.1}$$

which arises from solving boundary value problems of

$$-\Delta u - \sigma^2 u = f, \tag{1.2}$$

where σ^2 represents a positive shift and K and M represent the stiffness and the mass matrices respectively. Problems of this type also arise from solving generalized eigenvalue problem when an inverse shifted method is used. An extension of the dual-primal finite element tearing and interconnecting method (FETI-DP) was studied by Farhat and Li [19] for solving this class of indefinite problems, which arise from second-order forced elastic vibration problems. A key component in that extension is the use of a new coarse level problem which is based on free-space solutions of the Navier's homogeneous displacement equation of motion. This extension of FETI-DP algorithm has been shown numerically scalable by extensive experiments for solving this class of indefinite problems and has also been successfully extended to solving fourth-order and/or complex-valued problems, cf. [20, 16]. However, a convergence rate analysis of the FETI-DP algorithm for solving indefinite problems is still missing.

Our proposed BDDC algorithm in this paper for solving problems of the type (1.1), is motivated by the FETI-DP algorithm, where a coarse level problem containing free-space solutions of the inherent homogeneous PDE is also used in our algorithm to speed up the convergence. Our main contribution in this paper is the establishment of a convergence rate estimate for the proposed BDDC algorithm. It is proven that the convergence rate of the proposed BDDC depends polylogarithmically on the dimension of the individual subdomain problems and it improves with the decrease of the subdomain diameters, when the diameters of the subdomains are small enough and when an appropriate coarse level problem is used in the algorithm. We also show that the proposed BDDC method and the FETI-DP method for indefinite problems have the same eigenvalues except possibly those equal to 0 or 1, as for the symmetric positive definite cases. Our theory therefore also provides an insight on the convergence rate of the FETI-DP algorithm for solving indefinite problems. Numerical experiments in this paper show that the proposed BDDC algorithm has the same convergence rate as the FETI-DP method. One advantage of the BDDC algorithm, compared with the FETI-DP algorithm, is that it is easier to implement inexact subdomain and coarse level problem solvers in the algorithm, cf. [12, 31, 38, 50, 51].

In our BDDC preconditioner, a partially sub-assembled finite element problem is solved, for which only the coarse level, primal interface degrees of freedom are shared by neighboring subdomains. In our convergence rate analysis, an error bound for a solution of the indefinite problem by the partially sub-assembled finite element problem is crucial; we view that finite element problem as a non-conforming approximation of the indefinite problem. As in [8, 23], a perturbation approach is also used in our analysis to handle the indefiniteness of the problem. Similar approaches have also been used recently by the authors in the analysis of a BDDC algorithm for advection-diffusion problems [52].

This paper is organized as follows. The finite element discretization is given in Section 2. The decomposition of the domain and a partially sub-assembled finite element problem are discussed in Section 3. The BDDC and FETI-DP algorithms and their spectral equivalence are discussed in Section 4. In Section 5, a convergence rate analysis of the BDDC algorithm is given; the assumptions used in the proof are verified in Section 6. To conclude, numerical experiments are given in Section 7 to demonstrate the effectiveness of our method.

2. A finite element discretization. We consider the solution of the following partial differential equation on a bounded polyhedral domain $\Omega \in \mathbf{R}^d$, d = 2, 3,

$$\begin{cases} -\Delta u - \sigma^2 u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where σ is a real constant. The weak solution of (2.1) is given by: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall\, v \in H_0^1(\Omega), \tag{2.2}$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v - \sigma^2 uv$, and $(f, v) = \int_\Omega fv$. Under the assumption that (2.2) has a unique solution, we can prove the following regularity result for the weak solution.

LEMMA 2.1. *Let $\Omega \in \mathbf{R}^d$, $d = 2, 3$, be a bounded polyhedral domain with Lipschitz continuous boundary. Given any $f \in L_2(\Omega)$, let u be the unique solution of (2.2). Then $u \in H^{1+\gamma}(\Omega) \cap H_0^1(\Omega)$, for a certain $\gamma \in (1/2, 1]$, and there exists a positive constant C which is independent of σ such that*

$$\|u\|_{H^{1+\gamma}(\Omega)} \le C \left(1 + \frac{\sigma^2}{|\lambda_* - \sigma^2|}\right) \|f\|_{L_2(\Omega)},$$

where λ_* is the eigenvalue of the corresponding Laplace operator, closest to σ^2. In the case that $|\lambda_* - \sigma^2|$ is bounded away from zero, $\|u\|_{H^{1+\gamma}(\Omega)} \le C(1 + \sigma^2)\|f\|_{L_2(\Omega)}$, for a certain positive constant C independent of σ. The results hold for $\gamma = 1$, if Ω is convex.

Proof. Result for the case $\sigma = 0$ can be found in [25, Corollary 2.6.7]; see also [26, Section 9.1]. Here we give a proof for the case where $\sigma \neq 0$. We define an operator $\mathcal{K} : H_0^1(\Omega) \to H^{-1}(\Omega)$ by: given any $v \in H_0^1(\Omega)$,

$$\langle \mathcal{K}v, w\rangle = \int_\Omega \nabla v \cdot \nabla w, \quad \forall w \in H_0^1(\Omega).$$

Here $\langle \mathcal{K}v, w\rangle$ is the value of the functional $\mathcal{K}v$ at w; if $\mathcal{K}v \in L_2(\Omega)$ then $\langle \cdot, \cdot\rangle$ is the L_2 inner product. Given $f \in L_2(\Omega)$, let u be the unique solution of (2.2), i.e., $u \in H_0^1(\Omega)$ satisfies

$$\int_\Omega \nabla u \cdot \nabla w - \sigma^2 uw \;=\; \int_\Omega fw, \quad \forall\, w \in H_0^1(\Omega).$$

Then we have $\mathcal{K}u = f + \sigma^2 u$. Since $\mathcal{K}$ is invertible and its inverse $\mathcal{K}^{-1}$ is a map from the space $H^{-1}(\Omega)$ to $H_0^1(\Omega)$, we have $u = \mathcal{K}^{-1}(f + \sigma^2 u)$. Since $f \in L_2(\Omega)$ and $u \in H_0^1(\Omega)$, we know from the regularity result for the case $\sigma = 0$, cf. [26, Section 9.1], that $u \in H^{1+\gamma}(\Omega)$ for a certain $\gamma \in (1/2, 1]$, and $\|u\|_{H^{1+\gamma}(\Omega)} \leq C(\|f\|_{L_2(\Omega)} + \sigma^2\|u\|_{L_2(\Omega)})$ for a certain constant C independent of σ. If Ω is convex then the result holds for $\gamma = 1$.

To obtain a bound for $\|u\|_{L^2(\Omega)}$, we denote a set of orthonormal eigenfunction basis (in L_2 inner product) of the Laplace operator by ϕ_k, with corresponding eigenvalues λ_k, for $k = 1, 2, \ldots$. We denote the representations of u and f in this basis by

$$u = \sum_{k=1}^{\infty} u_k \phi_k, \quad \text{and} \quad f = \sum_{k=1}^{\infty} f_k \phi_k.$$

From (2.2), where taking $v = \phi_k$, for $k = 1, 2, \ldots$, we have

$$u_k = \frac{f_k}{\lambda_k - \sigma^2}.$$

Then we have

$$\|u\|_{L_2(\Omega)} = \sqrt{\sum_{k=1}^{\infty} u_k^2} = \sqrt{\sum_{k=1}^{\infty} \left(\frac{f_k}{\lambda_k - \sigma^2}\right)^2} \leq \frac{\|f\|_{L_2(\Omega)}}{|\lambda_* - \sigma^2|}.$$

Therefore

$$\|u\|_{H^{1+\gamma}(\Omega)} \leq C(\|f\|_{L_2(\Omega)} + \sigma^2\|u\|_{L_2(\Omega)}) = C\left(1 + \frac{\sigma^2}{|\lambda_* - \sigma^2|}\right)\|f\|_{L_2(\Omega)}.$$

□

In this paper we assume that σ^2 is bounded away from the eigenvalues of the Laplace operator such that the problem is well posed. Therefore, from Lemma 2.1, we always have $\|u\|_{H^{1+\gamma}(\Omega)} \leq C(1 + \sigma^2)\|f\|_{L_2(\Omega)}$, for a certain positive constant C independent of σ.

We consider a conforming finite element solution of (2.2). We denote the continuous piecewise linear finite element space by $\widehat{W}$. The finite element solution $u \in \widehat{W}$ satisfies

$$a(u, v) = (f, v), \quad \forall v \in \widehat{W}. \tag{2.3}$$

The resulting system of linear equations has the form

$$Au = (K - \sigma^2 M)u = f, \tag{2.4}$$

where K is the stiffness matrix, and M the mass matrix. In this paper, we will use the same notation u to denote a finite element function and its vector of coefficients with respect to the finite element basis functions. We will also use the same notation to denote the space of the finite element functions and the space of corresponding vectors, e.g., $\widehat{W}$. We have $|u|^2_{H^1(\Omega)} = u^T K u$, and $\|u\|^2_{L^2(\Omega)} = u^T M u$, for all $u \in \widehat{W}$.

We assume the finite element mesh is a union of shape regular elements with a typical element diameter h. We have the following standard approximation property of the finite element space $\widehat{W}$, cf. [48, Lemma B.6].

LEMMA 2.2. *There exists a constant C which is independent of the mesh size such that for all $u \in H^{1+\gamma}(\Omega)$, $\gamma \in (1/2, 1]$,*

$$\inf_{w \in \widehat{W}} \left(h|u - w|_{H^1(\Omega)} + \|u - w\|_{L_2(\Omega)}\right) \leq C h^{1+\gamma} |u|_{H^{1+\gamma}(\Omega)}.$$

3. A partially sub-assembled finite element space. A partially sub-assembled finite element space was introduced by Klawonn, Widlund, and Dryja [34] in a convergence analysis of the FETI-DP algorithm. It was later used by Li and Widlund [37, 38] to give an alternative formulation of the BDDC algorithm.

The domain Ω is decomposed into N nonoverlapping polyhedral subdomains Ω_i, $i = 1, 2, ..., N$. Each subdomain is a union of shape regular elements and the nodes on the boundaries of neighboring subdomains match across the interface $\Gamma = (\cup \partial\Omega_i) \backslash \partial\Omega$. The interface Γ is composed of subdomain faces and/or edges, which are regarded as open subsets of Γ, and of the subdomain vertices, which are end points of edges. In three dimensions, the subdomain faces are shared by two subdomains, and the edges typically by more than two; in two dimensions, each edge is shared by two subdomains. The interface of subdomain Ω_i is defined by $\Gamma_i = \partial\Omega_i \cap \Gamma$. We also denote the set of nodes on Γ_i by $\Gamma_{i,h}$. We note that all the algorithms considered here are well defined for the less regular subdomains that are obtained by mesh partitioners. When developing theory, we will assume, as is customary in domain decomposition theory, that each subdomain is the union of a bounded number of shape regular elements with diameters on the order of H; cf. [48, Section 4.2]. For recent results on the analysis for irregular subdomains in domain decomposition methods, see [14, 32].

The partially sub-assembled finite element space $\widetilde{W}$ is the direct sum of a coarse level primal subspace $\widehat{W}_\Pi$, of continuous coarse level finite element functions, and a dual space W_r, which is the product of local dual subspaces, i.e.,

$$\widetilde{W} = W_r \bigoplus \widehat{W}_\Pi = \left(\prod_{i=1}^{N} W_r^{(i)} \right) \bigoplus \widehat{W}_\Pi.$$

The space $\widehat{W}_\Pi$ corresponds to a few select subdomain interface degrees of freedom for each subdomain and is typically spanned by subdomain vertex nodal basis functions, and/or interface edge and/or face basis functions with weights at the nodes of the edge or face. These basis functions will correspond to the primal interface continuity constraints enforced in the BDDC and FETI-DP algorithms. For simplicity of our analysis, we will always assume that the basis has been changed so that we have explicit primal unknowns corresponding to the primal continuity constraints of edges or faces; these coarse level primal degrees of freedom are shared by neighboring subdomains. Each subdomain dual space $W_r^{(i)}$ corresponds to the subdomain interior and dual interface degrees of freedom and it is spanned by all the basis functions which vanish at the primal degrees of freedom. Thus, functions in the space $\widetilde{W}$ have a continuous coarse level, primal part and typically a discontinuous dual part across the subdomain interface.

REMARK 3.1. *As in many other papers on FETI-DP and BDDC algorithms, we talk about dual spaces. The discontinuity of elements of the dual spaces across the subdomain interface is controlled by using Lagrange multipliers in the FETI-DP algorithms.*

We define the bilinear form on the partially sub-assembled finite element space $\widetilde{W}$ by

$$\widetilde{a}(u, v) = \sum_{i=1}^{N} \int_{\Omega_i} \nabla u^{(i)} \cdot \nabla v^{(i)} - \sigma^2 u^{(i)} v^{(i)}, \quad \forall\, u, v \in \widetilde{W},$$

where $u^{(i)}$ and $v^{(i)}$ represent the restriction of u and v to subdomain Ω_i. The matrix corresponding to the bilinear form $\widetilde{a}(\cdot, \cdot)$ is denoted by $\widetilde{A}$. $\widetilde{A} = \widetilde{K} - \sigma^2 \widetilde{M}$, where $\widetilde{K}$

is the partially sub-assembled stiffness matrix and $\widetilde{M}$ is the partially sub-assembled mass matrix. We always assume that $\widetilde{A}$ is nonsingular, i.e., the following problem always has a unique solution: given any $g \in L_2(\Omega)$, find $u \in \widetilde{W}$ such that

$$\widetilde{a}(u,v) = (g,v), \quad \forall v \in \widetilde{W}. \tag{3.1}$$

We define partially sub-assembled norms on $\widetilde{W}$, by $\|w\|^2_{L_2(\Omega)} = \sum_{i=1}^N \|w^{(i)}\|^2_{L_2(\Omega_i)}$ and $|w|^2_{H^1(\Omega)} = \sum_{i=1}^N |w^{(i)}|^2_{H^1(\Omega_i)}$. In this paper $\|w\|_{L_2(\Omega)}$ and $|w|_{H^1(\Omega)}$, for functions $w \in \widetilde{W}$, always represent the corresponding partially sub-assembled norms. We also have, for any $w \in \widetilde{W}$, $|w|^2_{H^1(\Omega)} = w^T \widetilde{K} w$, and $\|w\|^2_{L_2(\Omega)} = w^T \widetilde{M} w$.

In our convergence analysis of the BDDC algorithms for solving the indefinite problems, we will establish an error bound for a solution of (2.1) by the partially sub-assembled finite element problem. For this purpose, we assume that in our decomposition of the global domain Ω, each subdomain Ω_i is of triangular or quadrilateral shape in two dimensions, and of tetrahedral or hexahedral shape in three dimensions. We also assume that the subdomains form a shape regular coarse mesh of Ω. We denote by $\widehat{W}_H$ the continuous linear, bilinear, or trilinear finite element space on the coarse subdomain mesh, and denote by I_H the finite element interpolation from the space $H^{1+\gamma}(\Omega)$, $\gamma \in (1/2, 1]$, to $\widehat{W}_H$. We have the following Bramble-Hilbert lemma; cf. [54, Theorem 2.3].

Lemma 3.2. *There exists a constant C which is independent of the mesh size such that for all $u \in H^{1+\gamma}(\Omega)$, $\gamma \in (1/2, 1]$, $\|u - I_H u\|_{H^{1+\gamma}(\Omega_i)} \leq C|u|_{H^{1+\gamma}(\Omega_i)}$, for $i = 1, 2, ..., N$.*

The problem matrix A in (2.4) can be obtained by assembling the partially sub-assembled problem matrix $\widetilde{A}$, i.e.,

$$A = \widetilde{R}^T \widetilde{A} \widetilde{R}, \tag{3.2}$$

where $\widetilde{R} : \widehat{W} \to \widetilde{W}$, is the injection operator from the space of continuous finite element functions to the space of partially sub-assembled finite element functions. In order to define a scaled injection operator, we need to introduce a positive scale factor $\delta_i^\dagger(x)$ for each node x on the interface Γ_i of subdomain Ω_i. In applications, these scale factors will depend on the heat conduction coefficient and the first of the Lamé parameters for scalar elliptic problems and the equations of linear elasticity, respectively; see [34, 33, 44]. Here, with $\mathcal{N}_x$ the set of indices of the subdomains which have x on their boundaries, we will only need to use inverse counting functions defined by $\delta_i^\dagger(x) = 1/card(\mathcal{N}_x)$, where $card(\mathcal{N}_x)$ is the number of the subdomains in the set $\mathcal{N}_x$. It is easy to see that $\sum_{j \in \mathcal{N}_x} \delta_j^\dagger(x) = 1$. Given these scale factors at the subdomain interface nodes, we can define the scaled injection operator $\widetilde{R}_D$; each row of $\widetilde{R}$ corresponds to a degree of freedom of the space $\widetilde{W}$, and multiplying each row which corresponds to a dual interface degree of freedom by the scale factor $\delta_i^\dagger(x)$, where $x \in \Gamma_{i,h}$ is the corresponding interface node, gives us $\widetilde{R}_D$.

4. A BDDC version of the FETI-DP method for indefinite problems.

4.1. A review of the FETI-DP method for indefinite problems. The FETI–DP method was first introduced for solving two-dimensional symmetric positive definite problems by Farhat, Lesoinne, LeTallec, Pierson, and Rixen [17]. In the FETI-DP algorithm, continuities of the primal variables at subdomain vertices are maintained (by subassembly) throughout the iteration, while other continuity constraints on the subdomain interface are enforced by Lagrange multipliers but only fully so until the convergence of the algorithm. It has been established, for both two and three dimensional problems, that the condition number of the FETI-DP algorithm is bounded by $C(1+\log(H/h))^2$, if the primal constraints and certain diagonal scalings are well chosen, cf. [40, 34, 33]. Here, C is a constant independent of the number of subdomains as well as the size of the elements.

In [19], the FETI-DP method was extended to solve a class of indefinite problems of the form (1.1), which arises from second-order forced elastic vibration problems. A key component in that extension is the use of a new coarse level problem which is based on free-space solutions of the Navier's homogeneous displacement equation of motion. This extension of FETI-DP algorithm has been shown numerically scalable by extensive experiments for solving this class of indefinite problems and has also been successfully extended to solving fourth-order and/or complex-valued problems, cf. [20, 16].

In the following, we review the FETI-DP algorithm [19] for solving the indefinite problem (2.4), which arises from the discretization of (2.1). Let us first look at some exact solutions of the homogeneous equation (2.1) (with $f = 0$) in free space. Denote by x the space coordinate vector, either in 2D or in 3D, and denote by θ any direction vector of unit length. All functions of the form

$$\cos(\sigma\theta \cdot x) \qquad \text{or} \qquad \sin(\sigma\theta \cdot x) \tag{4.1}$$

are solutions to the homogeneous equation (2.1) and they represent plane waves in the direction θ.

In the FETI-DP algorithm for solving (2.4), some coarse level primal continuity constraints corresponding to certain selected plane waves are enforced across the subdomain interface, i.e., the solution at each iteration step always has the same components corresponding to the chosen plane waves across the subdomain interface. Here we discuss how to enforce a plane wave continuity constraint for two-dimensional problems; the same approach can equally well be applied to three-dimensional problems, cf. [19]. Let $\mathcal{E}^{ij}$ be a subdomain interface edge, which is shared by two neighboring subdomains Ω_i and Ω_j. To define the coarse level finite element basis function corresponding to a plane wave, we denote by q the vector determined by the chosen plane wave restricted to $\mathcal{E}^{ij}$. We then choose the finite element function, which is determined by q at the nodes on $\mathcal{E}^{ij}$ and which vanishes elsewhere on the mesh, as a coarse level finite element basis function, i.e., we choose q as an element in the coarse level primal subspace $\widehat{W}_\Pi$ and change the finite element basis such that the other basis functions are orthogonal to q. By sharing this common coarse level primal degree of freedom between subdomains Ω_i and Ω_j, elements in the partially sub-assembled finite element space $\widetilde{W}$ always have a common component corresponding to q across $\mathcal{E}^{ij}$. In this paper, we always assume that the basis of the finite element space has been changed and that therefore there are explicit degrees of freedom corresponding to all the coarse level primal continuity constraints; for more details on the change of basis, see [37, 33, 30]. Another way of enforcing the continuity of primal variables across subdomain edges and faces without changing basis is by introducing additional Lagrange multipliers, cf. [18, 19].

REMARK 4.1. *On a subdomain interface edge or face, if the direction θ is chosen orthogonal to the edge or face, then the vector q corresponding to the plane wave $\cos(\sigma\theta \cdot x)$ is a vector of same components, which represents a constant average across the subdomain interface edge or face.*

REMARK 4.2. *We note that by choosing different directions θ in (4.1), differ-*

ent plane wave vectors can be obtained. Also both the cosine and sine modes of the plane waves can be included in the coarse level problem. By controlling the number of directions θ used, we control the number of the coarse level primal degrees of freedom related to the plane wave continuity constraints. As shown in [19], the larger the shift σ^2 in (2.4), the more directions need to be used to prevent a deterioration of the convergence rate.

To derive a formulation of the FETI-DP algorithm for indefinite problems, the partially sub-assembled problem matrix $\widetilde{A}$ is written in blocks corresponding to the subdomain interior variables and to the subdomain interface variables as

$$\widetilde{A} = \begin{bmatrix} A_{II} & \widetilde{A}_{I\Gamma} \\ \widetilde{A}_{\Gamma I} & \widetilde{A}_{\Gamma\Gamma} \end{bmatrix},$$

where A_{II} is block diagonal with one block for each subdomain, and $\widetilde{A}_{\Gamma\Gamma}$ is assembled across the subdomain interface Γ only with respect to the coarse level primal degrees of freedom.

The solution of the original system of linear equations (2.4) can be obtained by solving the following system of linear equations

$$\begin{bmatrix} A_{II} & \widetilde{A}_{I\Gamma} & 0 \\ \widetilde{A}_{\Gamma I} & \widetilde{A}_{\Gamma\Gamma} & B_\Gamma^T \\ 0 & B_\Gamma & 0 \end{bmatrix} \begin{bmatrix} u_I \\ \widetilde{u}_\Gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} f_I \\ \widetilde{f}_\Gamma \\ 0 \end{bmatrix}, \tag{4.2}$$

where $\widetilde{f}_\Gamma$ on the right hand side is assembled only with respect to the coarse level primal degrees of freedom across the subdomain interface. The matrix B_Γ has elements from the set $\{0, 1, -1\}$ and is used to enforce the continuity of the solution across the subdomain interface. Eliminating the variables u_I and $\widetilde{u}_\Gamma$ from (4.2), the following equation for the Lagrange multipliers λ is obtained,

$$B_\Gamma \widetilde{S}_\Gamma^{-1} B_\Gamma^T \lambda = B_\Gamma \widetilde{S}_\Gamma^{-1} (\widetilde{f}_\Gamma - \widetilde{A}_{\Gamma I} A_{II}^{-1} f_I), \tag{4.3}$$

where $\widetilde{S}_\Gamma = \widetilde{A}_{\Gamma\Gamma} - \widetilde{A}_{\Gamma I} A_{II}^{-1} \widetilde{A}_{I\Gamma}$.

In a FETI-DP algorithm for indefinite problems, a preconditioned GMRES iteration is used to solve (4.3). After obtaining the Lagrange multipliers λ, u_I and $\widetilde{u}_\Gamma$ can be found by back solving. In each iteration step, to multiply $\widetilde{S}_\Gamma^{-1}$ by a vector, a coarse level problem and subdomain Neumann boundary problems with fixed primal values need be solved. Two types of preconditioners have been used in FETI-DP algorithms: the Dirichlet preconditioner $B_{D,\Gamma} \widetilde{S}_\Gamma B_{D,\Gamma}^T$, and the lumped preconditioner $B_{D,\Gamma} \widetilde{A}_{\Gamma\Gamma} B_{D,\Gamma}^T$; cf. [21, 17]. Here $B_{D,\Gamma}$ is obtained from B_Γ by an appropriate scaling across the subdomain interface. In a Dirichlet preconditioner, to multiply $\widetilde{S}_\Gamma$ by a vector, subdomain Dirichlet boundary problems need be solved; cf. [17, 34].

When $\widetilde{S}_\Gamma$ is applied to a vector in the Dirichlet preconditioner, the operator $\widetilde{A}_{\Gamma\Gamma} - \widetilde{A}_{\Gamma I} A_{II}^{-1} \widetilde{A}_{I\Gamma}$ needs be multiplied by the vector. An alternative used in [19] is to multiply $\widetilde{A}_{\Gamma\Gamma} - \widetilde{K}_{\Gamma I} K_{II}^{-1} \widetilde{K}_{I\Gamma}$ in the Dirichlet preconditioner. This corresponds to the use of discrete harmonic extensions to the interior of subdomains. We denote this alternative Dirichlet preconditioner by $B_{D,\Gamma} \widetilde{S}_\Gamma^{\mathcal{H}} B_{D,\Gamma}^T$. Numerical experiments in Section 7 will show that using either $B_{D,\Gamma} \widetilde{S}_\Gamma B_{D,\Gamma}^T$ or $B_{D,\Gamma} \widetilde{S}_\Gamma^{\mathcal{H}} B_{D,\Gamma}^T$ gives almost the same convergence rate.

4.2. A BDDC version of the FETI-DP method for indefinite problems. We now present a BDDC version of the FETI-DP method for indefinite problems. The BDDC algorithms and the closely related primal versions of the FETI algorithms were proposed by Dohrmann [13], Fragakis and Papadrakakis [22], and Cros [11], for solving symmetric positive definite problems. Here we follow an alternative presentation of the BDDC algorithm given by Li and Widlund [38]. The formulation of BDDC preconditioners for the indefinite problems is in fact the same as for the symmetric positive definite case, except that the corresponding blocks are now indefinite matrices determined by $K - \sigma^2 M$ in (2.4).

A BDDC preconditioner for solving the indefinite problem (2.4) can be written as

$$B_1^{-1} = \widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D, \tag{4.4}$$

where $\widetilde{R}_D : \widehat{W} \rightarrow \widetilde{W}$, is the scaled restriction introduced in the end of Section 3. To multiply $\widetilde{A}^{-1}$ by a vector $\widetilde{g}$, the following partially sub-assembled problem needs be solved,

$$\widetilde{A}\widetilde{u} = \begin{bmatrix} A_{rr}^{(1)} & & & \widetilde{A}_{r\Pi}^{(1)} \\ & \ddots & & \vdots \\ & & A_{rr}^{(N)} & \widetilde{A}_{r\Pi}^{(N)} \\ \widetilde{A}_{\Pi r}^{(1)} & \dots & \widetilde{A}_{\Pi r}^{(N)} & \widetilde{A}_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} u_r^{(1)} \\ \vdots \\ u_r^{(N)} \\ u_\Pi \end{bmatrix} = \begin{bmatrix} g_r^{(1)} \\ \vdots \\ g_r^{(N)} \\ g_\Pi \end{bmatrix} = \widetilde{g}. \tag{4.5}$$

The leading diagonal blocks correspond to subdomain Neumann problems with given coarse level primal values. $\widetilde{A}_{\Pi\Pi}$ corresponds to the coarse level primal degrees of freedom and is assembled across the subdomain interface. $\widetilde{A}^{-1}\widetilde{g}$ is then computed by

$$\widetilde{A}^{-1}\widetilde{g} = \begin{bmatrix} A_{rr}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \widetilde{g} + \begin{bmatrix} -A_{rr}^{-1}\widetilde{A}_{r\Pi} \\ I \end{bmatrix} \widetilde{S}_\Pi^{-1} \begin{bmatrix} -\widetilde{A}_{\Pi r}A_{rr}^{-1} & I \end{bmatrix} \widetilde{g}, \tag{4.6}$$

where A_{rr}, $\widetilde{A}_{r\Pi}$, and $\widetilde{A}_{\Pi r}$ represent corresponding blocks of $\widetilde{A}$ in (4.5), $\widetilde{S}_\Pi = \widetilde{A}_{\Pi\Pi} - \sum_{i=1}^N \widetilde{A}_{\Pi r}^{(i)} {A_{rr}^{(i)}}^{-1} \widetilde{A}_{r\Pi}^{(i)}$, and which requires solving a coarse level problem and subdomain Neumann boundary problems with fixed primal values.

From (4.6), we see that the BDDC preconditioner (4.4) can be regarded as the summation of subdomain corrections and a coarse level correction. Let us denote

$$\Psi = \begin{bmatrix} -A_{rr}^{-1}\widetilde{A}_{r\Pi} \\ I \end{bmatrix}.$$

Then we can see that $\widetilde{S}_\Pi = \Psi^T \widetilde{A} \Psi$. Therefore the first term in the right hand of (4.6) corresponds to subdomain corrections for which all the coarse level primal variables vanish, and the second term corresponds to a correction projected on the coarse space determined by Ψ; cf. [39, 37]. Ψ represents extensions of the chosen coarse level finite element basis functions to the interior of subdomains, and these extensions are wavelike. In Figure 4.1, we plot the extensions of a subdomain corner basis function and a subdomain edge average basis function to the interior of the subdomain.

Another BDDC preconditioner for solving (2.4) is of the form

$$B_2^{-1} = (\widetilde{R}_D^T - \mathcal{H}J_D)\widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T). \tag{4.7}$$

FIG. 4.1. *Extension of the coarse level primal finite element basis functions to the interior of a subdomain: extension of a corner basis function (left); extension of an edge average basis function (right).*

Here $J_D : \widetilde{W} \to \widetilde{W}$. For any $w \in \widetilde{W}$, the component of $J_D w$ on subdomain Ω_i is defined by

$$(J_D w(x))^{(i)} = \sum_{j \in \mathcal{N}_x} \delta_j^\dagger(x)(w^{(i)}(x) - w^{(j)}(x)), \quad \forall x \in \Gamma_{i,h},$$

where $J_D w$ vanishes in the interior of the subdomain and for the coarse level primal component. For a matrix form of the operator J_D, see Toselli and Widlund [48, Section 6.3]. The component of $J_D^T w$ on subdomain Ω_i is then given by

$$(J_D^T w(x))^{(i)} = \sum_{j \in \mathcal{N}_x} (\delta_j^\dagger(x) w^{(i)}(x) - \delta_i^\dagger(x) w^{(j)}(x)), \quad \forall x \in \Gamma_{i,h}.$$

The subdomain interior and the coarse level primal components of $J_D^T w$ also vanish. The operator $\mathcal{H}$ in (4.7) is direct sum of the subdomain discrete harmonic extensions $\mathcal{H}^{(i)}$, where $\mathcal{H}^{(i)} = -K_{II}^{(i)-1} K_{I\Gamma}^{(i)}$, $i = 1, 2, ..., N$. $\mathcal{H} J_D$ represents the discrete harmonic extension of the jump of the dual interface variables to the interior of the subdomains.

An alternative to the discrete harmonic extension $\mathcal{H}$ used in the preconditioner B_2^{-1} is an extension based on solving the indefinite subdomain Dirichlet problems. Let $\mathcal{H}_A^{(i)} = -A_{II}^{(i)-1} A_{I\Gamma}^{(i)}$ and denote the direct sum of the $\mathcal{H}_A^{(i)}$ by $\mathcal{H}_A$. Then the corresponding preconditioner is defined by

$$B_3^{-1} = (\widetilde{R}_D^T - \mathcal{H}_A J_D)\widetilde{A}^{-1}(\widetilde{R}_D - J_D^T \mathcal{H}_A^T). \tag{4.8}$$

4.3. Spectral equivalence between the FETI-DP method and its BDDC counterpart for indefinite problems. Spectral equivalence results for FETI-DP and BDDC methods for symmetric positive definite problems were first proven by Mandel, Dohrmann, and Tezaur [39]; see also Fragakis and Papadrakakis [22], Li and Widlund [37, 38], and Brenner and Sung [6]. These arguments do not depend on the positive definiteness of the problem, and are also valid for indefinite problems; cf. [38, 37]. We have

THEOREM 4.3. *1. The preconditioned operator $B_1^{-1}A$ has the same eigenvalues as the preconditioned FETI-DP operator with the lumped preconditioner $B_{D,\Gamma}\widetilde{A}_{\Gamma\Gamma}B_{D,\Gamma}^T$, except for possible eigenvalues equal to 0 and 1.*

2. The preconditioned operator $B_3^{-1}A$ has the same eigenvalues as the preconditioned FETI-DP operator with the Dirichlet preconditioner $B_{D,\Gamma}\widetilde{S}_\Gamma B_{D,\Gamma}^T$, except for possible eigenvalues equal to 0 and 1.

We will demonstrate the spectral connection between the BDDC algorithms and the FETI-DP algorithms in Section 7. The spectral equivalence between the preconditioned BDDC operator $B_2^{-1}A$ and the preconditioned FETI-DP operator with preconditioner $B_{D,\Gamma}\widetilde{S}_{\Gamma}^{\mathcal{H}}B_{D,\Gamma}$ is not clear, even though their convergence rates are also quite similar in our numerical experiments.

5. Convergence rate analysis. The GMRES iteration is used in our BDDC algorithm to solve the preconditioned system of linear equations. For the convenience of our analysis, we use the inner product defined by the matrix $K + \sigma^2 M$ in the GMRES iteration. We define $\Lambda = K + \sigma^2 M$ and $\widetilde{\Lambda} = \widetilde{K} + \sigma^2 \widetilde{M}$, respectively. To estimate the convergence rate of the GMRES iteration, we use the following result due to Eisenstat, Elman, and Schultz [15].

THEOREM 5.1. *Let c_1 and C_2 be two parameters such that, for all $u \in \widehat{W}$,*

$$c_1 \langle u, u \rangle_\Lambda \leq \langle u, Tu \rangle_\Lambda , \tag{5.1}$$

$$\langle Tu, Tu \rangle_\Lambda \leq C_2 \langle u, u \rangle_\Lambda . \tag{5.2}$$

If $c_1 > 0$, then

$$\frac{\|r_m\|_\Lambda}{\|r_0\|_\Lambda} \leq \left(1 - \frac{c_1^2}{C_2}\right)^{m/2} ,$$

where r_m is the residual at step m of the GMRES iteration applied to the operator T.

REMARK 5.2. *The convergence rate of the GMRES iteration using the standard L_2 inner product will not be estimated in this paper. In our numerical experiments, we have found that using the $K+\sigma^2 M$ inner product or the standard L_2 inner product gives the same convergence rate. For a study of the convergence rates of the GMRES iteration for an additive Schwarz method in the Euclidean and energy norms, see Sarkis and Szyld [45].*

In Theorem 5.11, we will estimate c_1 and C_2 in (5.1) and (5.2), for the preconditioned BDDC operators $B_1^{-1}A$ and $B_2^{-1}A$. The analysis for $B_3^{-1}A$ is not available yet.

We first make the following assumption on the coarse level primal subspace $\widehat{W}_\Pi$ in our analysis.

ASSUMPTION 5.3. *The coarse level primal subspace $\widehat{W}_\Pi$ contains all subdomain corner degrees of freedom, one edge average degree of freedom on each edge corresponding to restriction of the plane wave $\cos(\sigma\theta \cdot x)$ on the edge with θ orthogonal to the edge, and, for three dimensional problems, one face average degree of freedom on each subdomain boundary face corresponding to restriction of the plane wave $\cos(\sigma\theta \cdot x)$ on the face with θ orthogonal to the face.*

Assumption 5.3 requires one coarse level primal degree of freedom for each edge and one for each face, respectively. Those constant edge or face average constraints correspond to the restriction of a cosine plane wave in (4.1) with the chosen angle θ perpendicular to the edge or to the face; see Remark 4.1. When more than one plane wave continuity constraints are enforced on the same edge or face, it can happen that the coarse level primal basis vectors are linearly dependent on that edge or face. In order to make sure that the primal basis functions maintain linear independence, we can use a singular value decomposition on each edge and face, in a preprocessing step of the algorithm, to single out those that are numerically linearly independent and should be retained in the coarse level primal subspace. This device of eliminating linearly dependent coarse level primal constraints has previously been applied in FETI-DP algorithms for solving indefinite problems, cf. [19].

Using Assumption 5.3, we have the following lemma, which is essentially a Poincaré-Friedrichs inequality proven by Brenner in [5, (1.3)].

LEMMA 5.4. *Let Assumption 5.3 hold. There exists a constant C, which is independent of H and h, such that* $\langle u,u\rangle_{\widetilde{M}} \le C\,\langle u,u\rangle_{\widetilde{K}}$, $\forall u \in \widetilde{W}$.

From Assumption 5.3, we also obtain a result on the stability of certain average operators, which are defined by $E_{D,1} = \widetilde{R}\widetilde{R}_D^T$ and $E_{D,2} = \widetilde{R}(\widetilde{R}_D^T - \mathcal{H}J_D)$, corresponding to the preconditioned BDDC operators $B_1^{-1}A$ and $B_2^{-1}A$, respectively. The following lemma can be found in [34, 33, 38].

LEMMA 5.5. *Let assumption 5.3 hold. Then there exist functions $\Phi_i(H,h)$, $i = 1,2$, such that*

$$|E_{D,i}w|^2_{H^1(\Omega)} \le \Phi_i(H,h)|w|^2_{H^1(\Omega)}, \quad \forall\ w \in \widetilde{W}.$$

where $\Phi_1(H,h) = CH/h$ and $\Phi_2(H,h) = C\left(1+\log(H/h)\right)^2$, for two-dimensional problems; $\Phi_1(H,h) = C(H/h)(1+\log{(H/h)})$ and $\Phi_2(H,h) = C\left(1+\log(H/h)\right)^2$, for three-dimensional problems. Here C is a positive constant independent of H and h.

Using Lemma 5.5, we can prove the following lemma.

LEMMA 5.6. *Let Assumption 5.3 hold. Then,*

$$\|E_{D,i}w\|^2_{\widetilde{\Lambda}} \le (1+C\sigma^2H^2)\Phi_i(H,h)\|w\|^2_{\widetilde{\Lambda}}, \quad \forall\ w \in \widetilde{W}, \quad i = 1,2,$$

where $\Phi_i(H,h)$ are determined as in Lemma 5.5, and C is a positive constant independent of σ, H, and h.

Proof. We know that

$$\|E_{D,i}w\|^2_{\widetilde{\Lambda}} = \|E_{D,i}w\|^2_{\widetilde{K}} + \sigma^2\|E_{D,i}w\|^2_{\widetilde{M}} = |E_{D,i}w|^2_{H^1(\Omega)} + \sigma^2\|E_{D,i}w\|^2_{L^2(\Omega)}.$$

Using Lemma 5.5 for the first term on the right side, and writing the second term as $E_{D,i}w = w - P_{D,i}w$, where $P_{D,i}$ represents a jump operator, cf. [38, 48, Lemma 6.10], we have

$$\|E_{D,i}w\|^2_{\widetilde{\Lambda}} \le \Phi_i(H,h)|w|^2_{H^1(\Omega)} + \sigma^2(\|w\|^2_{L^2(\Omega)} + \|P_{D,i}w\|^2_{L^2(\Omega)}).$$

From Assumption 5.3, we know that $P_{D,i}w$ has zero averages on the subdomain interfaces. Using a Poincaré-Friedrichs inequality and then a result similar to Lemma 5.5 on the stability of the jump operator $P_{D,i}$, cf. [34, Lemma 9], we have

$$\|P_{D,i}w\|^2_{L^2(\Omega)} \le CH^2|P_{D,i}w|^2_{H^1(\Omega)} \le CH^2\Phi_i(H,h)|w|^2_{H^1(\Omega)}.$$

Therefore, we have

$$\begin{aligned}\|E_{D,i}w\|^2_{\widetilde{\Lambda}} &\le \Phi_i(H,h)|w|^2_{H^1(\Omega)} + \sigma^2\|w\|^2_{L^2(\Omega)} + C\sigma^2H^2\Phi_i(H,h)|w|^2_{H^1(\Omega)}\\ &\le (1+C\sigma^2H^2)\Phi_i(H,h)(|w|^2_{H^1(\Omega)} + \sigma^2\|w\|^2_{L^2(\Omega)}).\end{aligned}$$

□

In the following, we define

$$C_L(H,h) = \begin{cases} (1+\log(H/h)), & \text{for three-dimensional problems,}\\ 1, & \text{for two-dimensional problems.}\end{cases} \tag{5.3}$$

The next assumption will be verified in Section 6.

ASSUMPTION 5.7. *There exists a positive constant C, which is independent of σ, H, and h, such that if $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then for all $u \in \widehat{W}$, $i = 1, 2$,*

$$\begin{aligned}\left|\left\langle w_i - \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{M}}\right| &\le C\frac{(1+\sigma^2)}{\sigma}H^\gamma C_L(H,h)(1+\sqrt{\Phi_i(H,h)})\left\langle u,u\right\rangle_\Lambda,\\ \left|\left\langle w_i - \widetilde{R}u, w_i\right\rangle_{\widetilde{M}}\right| &\le C\frac{(1+\sigma^2)}{\sigma}H^\gamma C_L(H,h)\left(1+\Phi_i(H,h)\right)\left\langle u,u\right\rangle_\Lambda,\\ \left|\left\langle z_i - \widetilde{R}u, w_i\right\rangle_{\widetilde{M}}\right| &\le C\frac{1}{\sigma}H\left(1+\Phi_i(H,h)\right)\left\langle u,u\right\rangle_\Lambda,\\ \|w_i\|_{\widetilde{\Lambda}} &\le C(1+\sqrt{\Phi_i(H,h)})\|u\|_\Lambda.\end{aligned}$$

Here $w_1 = \widetilde{A}^{-1}\widetilde{R}_DAu$, $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au$, $z_1 = \widetilde{M}^{-1}\widetilde{R}_DMu$, $z_2 = \widetilde{M}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Mu$, $\Phi_i(H,h)$ is determined in Lemma 5.5, and $C_L(H,h)$ is defined in (5.3).

The following lemma can be found in [48, Lemma B.31].

LEMMA 5.8. *The mass matrix $\widetilde{M}$ is spectrally equivalent to a diagonal matrix with diagonal entries on the order of h^d, where h is the mesh size and $d = 2, 3$, i.e., there exist positive constants c and C which are independent of the mesh size such that $ch^d \le \lambda_{min}(\widetilde{M}) \le \lambda_{max}(\widetilde{M}) \le Ch^d$.*

LEMMA 5.9. *There exists a positive constant C, which is independent of σ, H, and h, such that for all $u, v \in \widetilde{W}$, $|u^T\widetilde{A}v| \le C|u|_{\widetilde{\Lambda}}\,|v|_{\widetilde{\Lambda}}$, and $|u|_{\widetilde{\Lambda}} \le |u|_{\widetilde{K}} + \sigma|u|_{\widetilde{M}} \le \sqrt{2}|u|_{\widetilde{\Lambda}}$.*

Proof. To prove the first inequality, we have, for all $u, v \in \widetilde{W}$,

$$\begin{aligned}|u^T\widetilde{A}v| = |u^T\widetilde{K}v - \sigma^2u^T\widetilde{M}v| &\le (u^T\widetilde{K}u)^{1/2}(v^T\widetilde{K}v)^{1/2} + \sigma^2(u^T\widetilde{M}u)^{1/2}(v^T\widetilde{M}v)^{1/2}\\ &\le C(u^T\widetilde{\Lambda}u)^{1/2}(v^T\widetilde{\Lambda}v)^{1/2}.\end{aligned}$$

The second inequality can be derived by using $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ and $\sqrt{a} + \sqrt{b} \le \sqrt{2(a+b)}$, for any positive a and b. ∎

LEMMA 5.10. *Let Assumption 5.7 hold. Then there exists a constant C, which is independent of σ, H, and h, such that if $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then for $i = 1, 2$,*

$$\left\langle w_i, w_i\right\rangle_{\widetilde{\Lambda}} \le \left\langle u, B_i^{-1}Au\right\rangle_\Lambda + C\,\sigma(1+\sigma^2)H^\gamma C_L(H,h)\left(1+\Phi_i(H,h)\right)\left\langle u,u\right\rangle_\Lambda,\ \forall\, u \in \widehat{W},$$

where $w_1 = \widetilde{A}^{-1}\widetilde{R}_DAu$ and $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au$.

Proof. For $w_1 = \widetilde{A}^{-1}\widetilde{R}_DAu$, we have,

$$\begin{aligned}&\left\langle w_1, w_1\right\rangle_{\widetilde{\Lambda}} = \left\langle w_1, w_1\right\rangle_{\widetilde{A}} + 2\sigma^2\left\langle w_1, w_1\right\rangle_{\widetilde{M}}\\ &= u^TA\widetilde{R}_D^T\widetilde{A}^{-1}\widetilde{A}\widetilde{A}^{-1}\widetilde{R}_DAu + 2\sigma^2\left\langle w_1, w_1\right\rangle_{\widetilde{M}} = \left\langle u, B_1^{-1}Au\right\rangle_A + 2\sigma^2\left\langle w_1, w_1\right\rangle_{\widetilde{M}}\\ &= \left\langle u, B_1^{-1}Au\right\rangle_\Lambda - 2\sigma^2\left\langle u, B_1^{-1}Au\right\rangle_M + 2\sigma^2\left\langle w_1, w_1\right\rangle_{\widetilde{M}}\\ &= \left\langle u, B_1^{-1}Au\right\rangle_\Lambda - 2\sigma^2(u^TM\widetilde{R}_D^T\widetilde{A}^{-1}\widetilde{R}_DAu - \left\langle w_1, w_1\right\rangle_{\widetilde{M}})\\ &= \left\langle u, B_1^{-1}Au\right\rangle_\Lambda - 2\sigma^2(u^TM\widetilde{R}_D^T\widetilde{M}^{-1}\widetilde{M}\widetilde{A}^{-1}\widetilde{R}_DAu - \left\langle w_1, w_1\right\rangle_{\widetilde{M}})\\ &= \left\langle u, B_1^{-1}Au\right\rangle_\Lambda - 2\sigma^2\left\langle z_1 - w_1, w_1\right\rangle_{\widetilde{M}}\\ &= \left\langle u, B_1^{-1}Au\right\rangle_\Lambda - 2\sigma^2\left\langle z_1 - \widetilde{R}u, w_1\right\rangle_{\widetilde{M}} + 2\sigma^2\left\langle w_1 - \widetilde{R}u, w_1\right\rangle_{\widetilde{M}},\end{aligned}$$

where $z_1 = \widetilde{M}^{-1}\widetilde{R}_D M u$. For $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T \mathcal{H}^T)Au$, we have, cf. [38, Theorem 3],

$$\langle w_2, w_2\rangle_{\widetilde{\Lambda}} = \left\langle u, B_2^{-1}Au\right\rangle_\Lambda - 2\sigma^2\left\langle z_2 - \widetilde{R}u, w_2\right\rangle_{\widetilde{M}} + 2\sigma^2\left\langle w_2 - \widetilde{R}u, w_2\right\rangle_{\widetilde{M}},$$

where $z_2 = \widetilde{M}^{-1}(\widetilde{R}_D - J_D^T \mathcal{H}^T)Mu$. Then, using Assumption 5.7 for both cases, the lemma is proven. ∎

THEOREM 5.11. *Let Assumptions 5.3 and 5.7 hold. Let $T_i = B_i^{-1}A$ and $\Phi_i(H,h)$ be determined in Lemma 5.5. If $\sigma(1+\sigma^2)(1+\Phi_i(H,h))H^\gamma C_L(H,h)$ is sufficiently small, then, for $i = 1,2$,*

$$c\,\langle u,u\rangle_\Lambda \le \langle u, T_i u\rangle_\Lambda\,, \tag{5.4}$$

$$\langle T_i u, T_i u\rangle_\Lambda \le C(1+\sigma^2H^2)(1+\Phi_i(H,h)^2)\,\langle u,u\rangle_\Lambda\,, \tag{5.5}$$

where c and C are positive constants independent of σ, H, and h.

Proof. We only prove the result for the preconditioned operator $T_1 = B_1^{-1}A$. The few modifications in the proof for $B_2^{-1}A$ can be found in [38, Theorem 3].

We first prove the upper bound (5.5). Given any $u \in \widehat{W}$, let $w_1 = \widetilde{A}^{-1}\widetilde{R}_D A u$. We have,

$$\begin{aligned}
&\left\langle B_1^{-1}Au, B_1^{-1}Au\right\rangle_\Lambda = \left\langle \widetilde{R}_D^T\widetilde{A}^{-1}\widetilde{R}_D Au, \widetilde{R}_D^T\widetilde{A}^{-1}\widetilde{R}_D Au\right\rangle_\Lambda\\
&= \left\langle \widetilde{R}\widetilde{R}_D^T\widetilde{A}^{-1}\widetilde{R}_D Au, \widetilde{R}\widetilde{R}_D^T\widetilde{A}^{-1}\widetilde{R}_D Au\right\rangle_{\widetilde{\Lambda}} = \left\langle \widetilde{R}\widetilde{R}_D^T w_1, \widetilde{R}\widetilde{R}_D^T w_1\right\rangle_{\widetilde{\Lambda}} = \|E_{D,1}w_1\|_{\widetilde{\Lambda}}^2\\
&\le (1+C\sigma^2H^2)\Phi_1(H,h)\|w_1\|_{\widetilde{\Lambda}}^2 \le C(1+\sigma^2H^2)(1+\Phi_1(H,h)^2)\,\langle u,u\rangle_\Lambda\,,
\end{aligned}$$

where we have used Lemma 5.6 and the last inequality of Assumption 5.7.

To prove the lower bound (5.4), we have, from $\widetilde{R}^T\widetilde{R}_D = I$ and by using the Cauchy-Schwarz inequality, that

$$\begin{aligned}
\langle u,u\rangle_\Lambda &= \langle u,u\rangle_A + 2\sigma^2\langle u,u\rangle_M = u^TAu + 2\sigma^2\langle u,u\rangle_M\\
&= u^T\widetilde{R}^T\widetilde{A}\widetilde{A}^{-1}\widetilde{R}_D Au + 2\sigma^2\langle u,u\rangle_M = \left\langle w_1, \widetilde{R}u\right\rangle_{\widetilde{A}} + 2\sigma^2\langle u,u\rangle_M\\
&= \left\langle w_1, \widetilde{R}u\right\rangle_{\widetilde{\Lambda}} - 2\sigma^2\left\langle w_1, \widetilde{R}u\right\rangle_{\widetilde{M}} + 2\sigma^2\langle u,u\rangle_M\\
&= \left\langle w_1, \widetilde{R}u\right\rangle_{\widetilde{\Lambda}} - 2\sigma^2\left\langle w_1 - \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{M}}\\
&\le \langle w_1,w_1\rangle_{\widetilde{\Lambda}}^{1/2}\left\langle \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{\Lambda}}^{1/2} - 2\sigma^2\left\langle w_1 - \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{M}}\\
&= \langle w_1,w_1\rangle_{\widetilde{\Lambda}}^{1/2}\langle u,u\rangle_\Lambda^{1/2} - 2\sigma^2\left\langle w_1 - \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{M}}.
\end{aligned}$$

Then, from Assumption 5.7, we have

$$\langle u,u\rangle_\Lambda \le \langle w_1,w_1\rangle_{\widetilde{\Lambda}}^{1/2}\langle u,u\rangle_\Lambda^{1/2} + C\ \sigma(1+\sigma^2)H^\gamma C_L(H,h)(1+\sqrt{\Phi_1(H,h)})\,\langle u,u\rangle_\Lambda\,.$$

If $\sigma(1+\sigma^2)H^\gamma C_L(H,h)(1+\sqrt{\Phi_i(H,h)})$ is sufficiently small, then $\langle u,u\rangle_\Lambda \le C\,\langle w_1,w_1\rangle_{\widetilde{\Lambda}}$, where C is independent of σ, H and h. Therefore, using Lemma 5.10, we have

$$\langle u,u\rangle_\Lambda \le C\left(\left\langle u, B_1^{-1}Au\right\rangle_\Lambda + \sigma(1+\sigma^2)H^\gamma C_L(H,h)(1+\Phi_1(H,h))\,\langle u,u\rangle_\Lambda\right).$$

If $\sigma(1+\sigma^2)H^{\gamma}C_L(H,h)(1+\Phi_1(H,h))$ is sufficiently small, then (5.4) is proven with c independent of σ, H and h. ∎

Theorem 5.11 provides an estimate of the convergence rate of the BDDC algorithm for solving indefinite problems of the form (2.4), which depends on $\Phi_i(H,h)$ and on the product σH. For a fixed σ, the upper bound in (5.5) improves with a decrease of H.

6. Verifying Assumption 5.7. In this section, we give a proof of Assumption 5.7. We first prove an error bound in Lemmas 6.1-6.5 for the solution of the partially sub-assembled finite element problem.

Given $g \in L_2(\Omega)$, we define $\varphi_g \in H_0^1(\Omega)$ and $\widetilde{\varphi}_g \in \widetilde{W}$ as solutions to the following problems,

$$a(u,\varphi_g) = (u,g), \quad \forall u \in H_0^1(\Omega), \tag{6.1}$$

$$\widetilde{a}(w,\widetilde{\varphi}_g) = (w,g), \quad \forall w \in \widetilde{W}, \tag{6.2}$$

respectively. From Lemma 2.1, we know that $\varphi_g \in H_0^1(\Omega)\cap H^{1+\gamma}(\Omega)$, for some $\gamma \in (1/2,1]$.

LEMMA 6.1. *Let Assumption 5.3 hold. Given $g \in L_2(\Omega)$, let φ_g be the solution of (6.1). Let $L_h(\varphi_g,q) = (g,q) - \widetilde{a}(\varphi_g,q)$, for any $q \in \widetilde{W} \cup \left(H_0^1(\Omega)\cap H^{1+\gamma}(\Omega)\right)$, $\gamma \in (1/2,1]$. Then $|L_h(\varphi_g,q)| \le CH^{(1+\gamma)/2}C_L(H,h)|\varphi_g|_{H^{1+\gamma}(\Omega)}|q|_{H^1(\Omega)}$, where $C_L(H,h)$ is defined in (5.3).*

Proof. We give the proof only for the three-dimensional case; the two-dimensional case can be proved in a similar manner. Given any $q \in \widetilde{W} \cup \left(H_0^1(\Omega)\cap H^{1+\gamma}(\Omega)\right)$, we have

$$\begin{aligned} L_h(\varphi_g,q) &= (g,q) - \widetilde{a}(\varphi_g,q) = -\sum_{i=1}^N \int_{\Omega_i}\left(\nabla\varphi_g\nabla q - \sigma^2\varphi_g\; q\right)dx + \int_\Omega g\; qdx \\ &= -\sum_{i=1}^N \left(\int_{\partial\Omega_i}\partial_\nu\varphi_g q ds + \int_{\Omega_i}(-\Delta\varphi_g - \sigma^2\varphi_g)qdx\right) + \int_\Omega gqdx \\ &= -\sum_{i=1}^N \int_{\partial\Omega_i}\partial_\nu\varphi_g q ds = -\sum_{i=1}^N \sum_{\Gamma_{ij}\subset\partial\Omega_i}\int_{\Gamma_{ij}}\partial_\nu\varphi_g q ds, \end{aligned} \tag{6.3}$$

where we use the fact that $-\Delta\phi_g - \sigma^2\phi_g = g$ holds in the weak sense, cf. (6.1). Here Γ_{ij} represents the boundary faces of Ω_i.

From Assumption 5.3, we denote the common average of q on the face $\mathcal{F}^l$ of Γ_{ij} by $\overline{q}_{\mathcal{F}^l}$ and its common averages on the edges $\mathcal{E}^{lk}$ by $\overline{q}_{\mathcal{E}^{lk}}$. For each subdomain interface edge $\mathcal{E}^{lk}$, let $\vartheta_{\mathcal{E}^{lk}}$ be the standard finite element edge cut-off function which vanishes at all interface nodes except those of the edge $\mathcal{E}^{lk}$ where it takes the value 1. For three-dimensional problems, we denote the finite element face cut-off functions by $\vartheta_{\mathcal{F}^l}$, which vanishes at all interface nodes except those of $\mathcal{F}^l$ where it takes the value 1. Let I_h be the interpolation operator into the finite element space. Since the finite element cut-off functions $\vartheta_{\mathcal{F}^l}$ and $\vartheta_{\mathcal{E}^{lk}}$ provide a partition of unity, cf. [48, Section 4.6], we have, from (6.3),

$$L_h(\varphi_g,q) \;=\; -\sum_{i=1}^N \sum_{\Gamma_{ij}\subset\partial\Omega_i}\left\{\int_{\mathcal{F}^l}\left(\partial_\nu\varphi_g I_h(\vartheta_{\mathcal{F}^l}(q-\overline{q}_{\mathcal{F}^l}))\right)ds\right.$$

$$
\begin{aligned}
&\quad + \sum_{\mathcal{E}^{lk}\subset\Gamma_{ij}} \int_{\mathcal{F}^l} \partial_\nu \varphi_g I_h(\vartheta_{\mathcal{E}^{lk}}(q-\overline{q}_{\mathcal{E}^{lk}}))ds\Big\}, \\
&= -\sum_{i=1}^{N}\sum_{\Gamma_{ij}\subset\partial\Omega_i} \Big\{ \int_{\mathcal{F}^l} (\partial_\nu(\varphi_g - I_H\varphi_g) I_h(\vartheta_{\mathcal{F}^l}(q-\overline{q}_{\mathcal{F}^l})))\, ds \\
&\quad + \sum_{\mathcal{E}^{lk}\subset\Gamma_{ij}} \int_{\mathcal{F}^l} \partial_\nu \varphi_g I_h(\vartheta_{\mathcal{E}^{lk}}(q-\overline{q}_{\mathcal{E}^{lk}}))ds\Big\}, \\
&:= I_1 + I_2 \qquad (6.4)
\end{aligned}
$$

where $I_H\varphi_g$ represents the interpolation of φ_g into the space $\widehat{W}_H$ on the coarse subdomain mesh. The terms in (6.4) are bounded as follows.

From the Cauchy-Schwarz inequality, we have for the first term that

$$
|I_1| \le \sum_{i=1}^{N}\sum_{\Gamma_{ij}\subset\partial\Omega} \left(\int_{\mathcal{F}^l} |\nabla(\varphi_g - I_H\varphi_g)|^2 \, ds \int_{\mathcal{F}^l} |I_h(\vartheta_{\mathcal{F}^l}(q^{(i)} - \overline{q}_{\mathcal{F}^l}))|^2 \, ds \right)^{1/2}. \qquad (6.5)
$$

Using a trace theorem and Lemma 3.2, we have for the first factor

$$
\begin{aligned}
\int_{\mathcal{F}^l} |\nabla(\varphi_g - I_H\varphi_g)|^2 \, ds &\le CH^\gamma \|\nabla(\varphi_g - I_H\varphi_g)\|^2_{H^\gamma(\Omega_i)} \\
&\le CH^\gamma \|\varphi_g - I_H\varphi_g\|^2_{H^{1+\gamma}(\Omega_i)} \le CH^\gamma |\varphi_g|^2_{H^{1+\gamma}(\Omega_i)}. \qquad (6.6)
\end{aligned}
$$

For the second factor in (6.5) , we have

$$
\begin{aligned}
&\int_{\mathcal{F}^l} |I_h(\vartheta_{\mathcal{F}^l}(q-\overline{q}_{\mathcal{F}^l}))|^2 \, ds \le CH \|I_h(\vartheta_{\mathcal{F}^l}(q-\overline{q}_{\mathcal{F}^l}))\|^2_{H^1(\Omega_i)} \\
&\le CH(1+\log\frac{H}{h})^2 \|q-\overline{q}_{\mathcal{F}^l}\|^2_{H^1(\Omega_i)} \le CH(1+\log\frac{H}{h})^2 |q|^2_{H^1(\Omega_i)}, \qquad (6.7)
\end{aligned}
$$

where we have used a trace theorem for the first step, a Poincaré-Friedrichs inequality and [48, Lemma 4.24] in the second, and a Poincaré-Friedrichs inequality in the last step. Combining (6.5), (6.6), and (6.7), we have the following bound for I_1,

$$
\begin{aligned}
|I_1| &\le CH^{(1+\gamma)/2}\left(1+\log\frac{H}{h}\right)\sum_{i=1}^{N} |\varphi_g|_{H^{1+\gamma}(\Omega_i)} |q|_{H^1(\Omega_i)} \\
&\le CH^{(1+\gamma)/2}\left(1+\log\frac{H}{h}\right) |\varphi_g|_{H^{1+\gamma}(\Omega)} |q|_{H^1(\Omega)}.
\end{aligned}
$$

The estimate for I_2 is similar to the estimate for I_1. Instead of using (6.6) and (6.7), we have, by using a trace theorem,

$$
\int_{\mathcal{F}^l} |\nabla\varphi_g|^2 \, ds \le CH^\gamma \|\nabla\varphi_g\|^2_{H^\gamma(\Omega_i)} \le CH^\gamma \|\varphi_g\|^2_{H^{1+\gamma}(\Omega_i)}, \qquad (6.8)
$$

and

$$
\begin{aligned}
&\int_{\mathcal{F}^l} |I_h\vartheta_{\mathcal{E}^{lk}}(q-\overline{q}_{\mathcal{E}^{lk}})|^2 \, ds \le Ch \|I_h\vartheta_{\mathcal{E}^{lk}}(q-\overline{q}_{\mathcal{E}^{lk}})\|^2_{L^2(\mathcal{E}^{lk})} \qquad (6.9) \\
&\le Ch(1+\log\frac{H}{h}) \|I_h\vartheta_{\mathcal{E}^{lk}}(q-\overline{q}_{\mathcal{E}^{lk}})\|^2_{H^1(\Omega_i)} \le Ch(1+\log\frac{H}{h})^2 |q|^2_{H^1(\Omega_i)}.
\end{aligned}
$$

In the first step of (6.9), we use the fact that $I_h\vartheta_{\mathcal{E}^{lk}}(q-\overline{q}_{\mathcal{E}^{lk}})$ is different from zero only in the strip of elements next to the edge $\mathcal{E}^{lk}$; in the second and the last steps, we use [48, Lemma 4.16], [48, Corollary 4.20], and a Poincaré-Friedrichs inequality. Combining (6.8) and (6.9), we have

$$\begin{aligned}|I_2| &\le Ch^{1/2}H^{\gamma/2}\left(1+\log\frac{H}{h}\right)\sum_{i=1}^{N}\|\varphi_g\|_{H^{1+\gamma}(\Omega_i)}|q|_{H^1(\Omega_i)}\\ &\le CH^{(1+\gamma)/2}\left(1+\log\frac{H}{h}\right)|\varphi_g|_{H^{1+\gamma}(\Omega)}|q|_{H^1(\Omega)}.\end{aligned}$$

□

REMARK 6.2. *In the case of two-dimensional problems, (6.4) becomes*

$$L_h(q,\varphi_g)=\sum_{i=1}^{N}\sum_{\mathcal{E}^{ij}\subset\partial\Omega_i}\int_{\mathcal{E}^{ij}}(\partial_n\varphi_g(q-\overline{q}_{\mathcal{E}^{ij}}))\,ds,$$

where the finite element cut-off functions are no longer used. The bound for $L_h(q,\varphi_g)$ then follows from the similar steps as for the bounds of I_1 in the proof, but applied on the edges, where the logarithmic factor related to the use of finite element cut-off functions disappears.

The following lemma is established by using Lemma 6.1.

LEMMA 6.3. *Let Assumption 5.3 hold. Given $g\in L_2(\Omega)$, let φ_g and $\widetilde{\varphi}_g$ be solutions to problems (6.1) and (6.2), respectively. If $\sigma(1+\sigma^2)h^\gamma$ is sufficiently small, then*

$$\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)}\le C(1+\sigma^2)H^\gamma C_L(H,h)\left(|\varphi_g-\widetilde{\varphi}_g|_{H^1(\Omega)}+H^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}(\Omega)}\right),$$

where C is a positive constant independent of σ, H, and h, and $C_L(H,h)$ is defined in (5.3).

Proof. Given any $q\in L_2(\Omega)$, let $z_q\in H_0^1(\Omega)$ be the solution of

$$a(z_q,v)=(q,v),\quad \forall v\in H_0^1(\Omega). \tag{6.10}$$

We know, from Lemma 2.1, that $z_q\in H^{1+\gamma}(\Omega)$ and $\|z_q\|_{H^{1+\gamma}(\Omega)}\le C(1+\sigma^2)\|q\|_{L^2(\Omega)}$, for some $\gamma\in(1/2,1]$. From a Strang lemma, cf. [10, Remark 31.1], we have

$$\begin{aligned}\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)}\le \sup_{q\in L_2(\Omega)}\frac{1}{\|q\|_{L^2(\Omega)}}\Big\{&\inf_{\widetilde{z}\in\widetilde{W}}\left(|a_h(\varphi_g-\widetilde{\varphi}_g,z_q-\widetilde{z})|\right.\\ &\left.+|L_h(\varphi_g,z_q-\widetilde{z})|+|L_h(z_q,\varphi_g-\widetilde{\varphi}_g)|\ \right)\Big\}.\end{aligned}$$

Then, using Lemmas 5.9 and 6.1, we have

$$\begin{aligned}\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)}\le C\sup_{q\in L_2(\Omega)}\frac{1}{\|q\|_{L^2(\Omega)}}\inf_{\widetilde{z}\in\widetilde{W}}&\left\{(|\varphi_g-\widetilde{\varphi}_g|_{H^1(\Omega)}+\sigma\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)})\right.\\ (|z_q-\widetilde{z}|_{H^1(\Omega)}+\sigma\|z_q-\widetilde{z}\|_{L^2(\Omega)})&+H^{(1+\gamma)/2}C_L(H,h)\left(|\varphi_g|_{H^{1+\gamma}}|z_q-\widetilde{z}|_{H^1(\Omega)}\right.\\ &\left.\left.+|z_q|_{H^{1+\gamma}}|\varphi_g-\widetilde{\varphi}_g|_{H^1(\Omega)}\right)\right\}.\end{aligned}$$

From Lemma 2.2, we know $\inf_{\widetilde{z}\in\widetilde{W}}|z_q-\widetilde{z}|_{H^1(\Omega)}\le Ch^\gamma\|z_q\|_{H^{1+\gamma}(\Omega)}$, and $\inf_{\widetilde{z}\in\widetilde{W}}\|z_q-\widetilde{z}\|_{L_2(\Omega)}\le Ch^{1+\gamma}\|z_q\|_{H^{1+\gamma}(\Omega)}$. Then from $\|z_q\|_{H^{1+\gamma}(\Omega)}\le C(1+\sigma^2)\|q\|_{L^2(\Omega)}$, we have

$$\begin{aligned}\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)}\le C\sigma(1+\sigma^2)h^\gamma(1+\sigma h)\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)}+C(1+\sigma^2)H^\gamma C_L(H,h)\\ \left((1+H^{(1-\gamma)/2}+\sigma h)|\varphi_g-\widetilde{\varphi}_g|_{H^1(\Omega)}+H^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}(\Omega)}\right).\end{aligned}$$

If $\sigma(1+\sigma^2)h^\gamma$ is sufficiently small and therefore σh is less than a certain constant, then we have $\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)} \leq C(1+\sigma^2)H^\gamma C_L(H,h)(|\varphi_g-\widetilde{\varphi}_g|_{H^1(\Omega)}+H^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}})$. ∎

The proof of the following lemma is essentially an extension of the proof in [2, Chapter III, Lemma 1.2] to the indefinite case.

Lemma 6.4. *Let Assumption 5.3 hold. Given $g \in L_2(\Omega)$, let φ_g and $\widetilde{\varphi}_g$ be solutions to problems (6.1) and (6.2), respectively. If σh is less than a certain constant, then*

$$|\varphi_g-\widetilde{\varphi}_g|_{H^1(\Omega)} \leq \sigma\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)} + CH^\gamma C_L(H,h)|\varphi_g|_{H^{1+\gamma}(\Omega)},$$

where C is a positive constant independent of σ, H, and h.

Proof. For any given $\widetilde{\psi} \in \widetilde{W}$, we have

$$\begin{aligned}|\widetilde{\varphi}_g-\widetilde{\psi}|^2_{H^1(\Omega)} - \sigma^2\|\widetilde{\varphi}_g-\widetilde{\psi}\|^2_{L^2(\Omega)} &= \widetilde{a}(\widetilde{\varphi}_g-\widetilde{\psi},\widetilde{\varphi}_g-\widetilde{\psi})\\ &= \widetilde{a}(\varphi_g-\widetilde{\psi},\widetilde{\varphi}_g-\widetilde{\psi}) + ((g,\widetilde{\varphi}_g-\widetilde{\psi}) - \widetilde{a}(\varphi_g,\widetilde{\varphi}_g-\widetilde{\psi})).\end{aligned}$$

Dividing by $|\widetilde{\varphi}_g-\widetilde{\psi}|_{H^1(\Omega)}+\sigma\|\widetilde{\varphi}_g-\widetilde{\psi}\|_{L^2(\Omega)}$ on both sides and denoting $\widetilde{\varphi}_g-\widetilde{\psi}$ by q_h, we have, from Lemmas 5.9 and 6.1, that

$$\begin{aligned}&|\widetilde{\varphi}_g-\widetilde{\psi}|_{H^1(\Omega)} - \sigma\|\widetilde{\varphi}_g-\widetilde{\psi}\|_{L^2(\Omega)}\\ &\leq (|\varphi_g-\widetilde{\psi}|_{H^1(\Omega)}+\sigma\|\varphi_g-\widetilde{\psi}\|_{L^2(\Omega)}) + \frac{|(g,q_h)-\widetilde{a}(\varphi_g,q_h)|}{|q_h|_{H^1(\Omega)}}\\ &\leq |\varphi_g-\widetilde{\psi}|_{H^1(\Omega)}+\sigma\|\varphi_g-\widetilde{\psi}\|_{L^2(\Omega)} + CH^{(1+\gamma)/2}C_L(H,h)|\varphi_g|_{H^{1+\gamma}(\Omega)}.\end{aligned}$$

Using the triangle inequality, we have

$$\begin{aligned}|\varphi_g-\widetilde{\varphi}_g|_{H^1(\Omega)} &\leq |\widetilde{\varphi}_g-\widetilde{\psi}|_{H^1(\Omega)} + |\varphi_g-\widetilde{\psi}|_{H^1(\Omega)}\\ &\leq \sigma\|\widetilde{\varphi}_g-\widetilde{\psi}\|_{L^2(\Omega)} + 2|\varphi_g-\widetilde{\psi}|_{H^1(\Omega)} + \sigma\|\varphi_g-\widetilde{\psi}\|_{L^2(\Omega)}\\ &\quad + CH^{(1+\gamma)/2}C_L(H,h)|\varphi_g|_{H^{1+\gamma}(\Omega)}\\ &\leq \sigma\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)} + 2|\varphi_g-\widetilde{\psi}|_{H^1(\Omega)} + 2\sigma\|\varphi_g-\widetilde{\psi}\|_{L^2(\Omega)}\\ &\quad + CH^{(1+\gamma)/2}C_L(H,h)|\varphi_g|_{H^{1+\gamma}(\Omega)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}|\varphi_g-\widetilde{\varphi}_g|_{H^1(\Omega)} &\leq \sigma\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)} + 2\inf_{\widetilde{\psi}\in\widetilde{W}}(|\varphi_g-\widetilde{\psi}|_{H^1(\Omega)}+\sigma\|\varphi_g-\widetilde{\psi}\|_{L^2(\Omega)})\\ &\quad + CH^{(1+\gamma)/2}C_L(H,h)|\varphi_g|_{H^{1+\gamma}(\Omega)}\\ &\leq \sigma\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)} + CH^\gamma C_L(H,h)|\varphi_g|_{H^{1+\gamma}(\Omega)},\end{aligned}$$

where in the last step, we have used the approximation property in Lemma 2.2 and that σh is less than a certain constant. ∎

The following lemma follows from Lemmas 6.3 and 6.4.

Lemma 6.5. *Let Assumption 5.3 hold. Given $g \in L_2(\Omega)$, let φ_g and $\widetilde{\varphi}_g$ be solutions to problems (6.1) and (6.2), respectively. If $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then*

$$|\varphi_g-\widetilde{\varphi}_g|_{H^1(\Omega)} + \sigma\|\varphi_g-\widetilde{\varphi}_g\|_{L^2(\Omega)} \leq C(1+\sigma^2)H^\gamma C_L(H,h)\|g\|_{L^2(\Omega)},$$

where C is a positive constant independent of σ, H, and h.

Proof. From Lemmas 6.4 and 6.3, we have

$$\begin{aligned}
&|\varphi_g - \widetilde{\varphi}_g|_{H^1(\Omega)} + \sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L^2(\Omega)} \leq 2\sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L^2(\Omega)} + CH^\gamma C_L(H,h)|\varphi_g|_{H^{1+\gamma}(\Omega)} \\
&\leq C_1\sigma(1+\sigma^2)H^\gamma C_L(H,h)(|\varphi_g - \widetilde{\varphi}_g|_{H^1(\Omega)} + H^{(1+\gamma)/2}|\varphi_g|_{H^{1+\gamma}(\Omega)}) \\
&\quad + CH^\gamma C_L(H,h)|\varphi_g|_{H^{1+\gamma}(\Omega)}.
\end{aligned}$$

Therefore, if $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then we have

$$\begin{aligned}
|\varphi_g - \widetilde{\varphi}_g|_{H^1(\Omega)} + \sigma\|\varphi_g - \widetilde{\varphi}_g\|_{L^2(\Omega)} &\leq CH^\gamma C_L(H,h)|\varphi_g|_{H^{1+\gamma}} \\
&\leq C(1+\sigma^2)H^\gamma C_L(H,h)\|g\|_{L^2(\Omega)}.
\end{aligned}$$

□

LEMMA 6.6. *Let Assumption 5.3 hold. Given $u \in \widehat{W}$, let $w_1 = \widetilde{A}^{-1}\widetilde{R}_D A u$, and $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au$. If $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then*

$$\|w_i - u\|_{L^2(\Omega)} \leq C(1+\sigma^2)H^\gamma C_L(H,h)(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_\Lambda), \quad i = 1,2,$$

where C is a positive constant which is independent of σ, H, and h.

Proof. Given $u \in \widehat{W}$, let $w_1 = \widetilde{A}^{-1}\widetilde{R}_D Au$. We have, for any $v \in \widetilde{W}$,

$$v^T\widetilde{A}w_1 = v^T\widetilde{A}\widetilde{A}^{-1}\widetilde{R}_D Au = v^T\widetilde{A}\widetilde{A}^{-1}\widetilde{R}_D\widetilde{R}^T\widetilde{A}\widetilde{R}u = \left\langle \widetilde{R}u, \widetilde{R}\widetilde{R}_D^T v\right\rangle_{\widetilde{A}}.$$

Let $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au$. We have, for any $v \in \widetilde{W}$,

$$v^T\widetilde{A}w_2 = v^T\widetilde{A}\widetilde{A}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Au = \left\langle \widetilde{R}u, \widetilde{R}(\widetilde{R}_D^T - \mathcal{H}J_D)v\right\rangle_{\widetilde{A}}.$$

This shows that for any $u \in \widehat{W}$,

$$\widetilde{a}(w_i, v) = \widetilde{a}(\widetilde{R}u, E_{D,i}v), \quad \forall v \in \widetilde{W},\ i = 1,2, \tag{6.11}$$

where $E_{D,1} = \widetilde{R}\widetilde{R}_D^T$ and $E_{D,2} = \widetilde{R}(\widetilde{R}_D^T - \mathcal{H}J_D)$. Therefore,

$$\widetilde{a}(w_i - \widetilde{R}u, v) = 0, \quad \forall v \in \widetilde{R}(\widehat{W}),\ i = 1,2. \tag{6.12}$$

For any $g \in L_2(\Omega)$, let φ_g and $\widetilde{\varphi}_g$ be solutions to problems (6.1) and (6.2), respectively. We denote by $I_h\varphi_g$ the finite element interpolation of φ_g onto the space $\widehat{W}$. From (6.1) and (6.2), we have,

$$\begin{aligned}
(w_i - u, g) = (w_i, g) - (u, g) &= \widetilde{a}(w_i, \widetilde{\varphi}_g) - a(u, \varphi_g) \\
&= \widetilde{a}(w_i, \widetilde{\varphi}_g) - a(u, I_h\varphi_g) - a(u, \varphi_g - I_h\varphi_g) \\
&= \widetilde{a}(w_i, \widetilde{\varphi}_g) - \widetilde{a}(\widetilde{R}u, \widetilde{R}I_h\varphi_g) - a(u, \varphi_g - I_h\varphi_g) \\
&= \widetilde{a}(w_i - \widetilde{R}u, \widetilde{\varphi}_g) - \widetilde{a}(\widetilde{R}u, \widetilde{R}I_h\varphi_g - \widetilde{\varphi}_g) - a(u, \varphi_g - I_h\varphi_g).
\end{aligned}$$

From (6.12), we know that $\widetilde{a}(w_i - \widetilde{R}u, \widetilde{R}I_h\varphi_g) = 0$. Therefore,

$$\begin{aligned}
|(w_i - u, g)| &= |\widetilde{a}(w_i - \widetilde{R}u, \widetilde{\varphi}_g - \widetilde{R}I_h\varphi_g) - \widetilde{a}(\widetilde{R}u, \widetilde{R}I_h\varphi_g - \widetilde{\varphi}_g) - a(u, \varphi_g - I_h\varphi_g)| \\
&\leq C(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|\widetilde{R}u\|_{\widetilde{\Lambda}})(\|\widetilde{\varphi}_g - \widetilde{R}I_h\varphi_g\|_{\widetilde{\Lambda}} + \|\varphi_g - I_h\varphi_g\|_\Lambda) \\
&\leq C(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|\widetilde{R}u\|_{\widetilde{\Lambda}})(|\widetilde{\varphi}_g - I_h\varphi_g|_{H^1(\Omega)} + \sigma\|\widetilde{\varphi}_g - I_h\varphi_g\|_{L^2(\Omega)} + \\
&\qquad |\varphi_g - I_h\varphi_g|_{H^1(\Omega)} + \sigma\|\varphi_g - I_h\varphi_g\|_{L^2(\Omega)}) \\
&\leq C(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|\widetilde{R}u\|_{\widetilde{\Lambda}})(|\widetilde{\varphi}_g - \varphi_g|_{H^1(\Omega)} + \sigma\|\widetilde{\varphi}_g - \varphi_g\|_{L^2(\Omega)} + \\
&\qquad 2|\varphi_g - I_h\varphi_g|_{H^1(\Omega)} + 2\sigma\|\varphi_g - I_h\varphi_g\|_{L^2(\Omega)})
\end{aligned}$$

where we have used Lemma 5.9 in the middle. Then, using Lemmas 6.5 and 2.2, we have that if $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then

$$\left|(w_i - u, g)\right| \le C(1+\sigma^2)H^\gamma C_L(H,h)(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_\Lambda)\|g\|_{L^2(\Omega)}.$$

Therefore,

$$\|w_i - u\|_{L^2(\Omega)} = \sup_{g\in L_2(\Omega)} \frac{\left|(w_i - u, g)\right|}{\|g\|_{L^2(\Omega)}} \le C(1+\sigma^2)H^\gamma C_L(H,h)(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_\Lambda).$$

□

Lemma 6.7. *Let Assumption 5.3 hold. Given $u \in \widehat{W}$, let $w_1 = \widetilde{A}^{-1}\widetilde{R}_D A u$, and $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T \mathcal{H}^T)Au$. If $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then*

$$\|w_i\|_{\widetilde{K}} \le C(1+\sqrt{\Phi_i(H,h)})\|u\|_\Lambda, \quad i = 1, 2,$$

where $\Phi_i(H,h)$ are determined as in Lemma 5.5, and C is a positive constant independent of σ, H, and h.

Proof. We have, for any $w \in \widetilde{W}$, $\|w\|^2_{\widetilde{K}} - \sigma^2\|w\|^2_{\widetilde{M}} = \widetilde{a}(w,w)$. Then, from (6.11) and Lemma 5.9, we have, for $i = 1, 2$,

$$\|w_i\|^2_{\widetilde{K}} - \sigma^2\|w_i\|^2_{\widetilde{M}} = \widetilde{a}(w_i, w_i) = \widetilde{a}(\widetilde{R}u, E_{D,i}w_i) \le \|u\|_\Lambda \|E_{D,i}w_i\|_{\widetilde{\Lambda}}.$$

Using Lemma 5.6, we have that if σH is less than a certain constant, then

$$\|w_i\|^2_{\widetilde{K}} - \sigma^2\|w_i\|^2_{\widetilde{M}} \le C\sqrt{\Phi_i(H,h)}\|u\|_\Lambda \|w_i\|_{\widetilde{\Lambda}}, \quad i = 1, 2.$$

Dividing by $\|w_i\|_{\widetilde{K}} + \sigma\|w_i\|_{\widetilde{M}}$ on both sides and using Lemma 5.9, we have,

$$\|w_i\|_{\widetilde{K}} - \sigma\|w_i\|_{\widetilde{M}} \le C\sqrt{\Phi_i(H,h)}\|u\|_\Lambda.$$

Using this inequality and the triangle inequality, in particular that $-\|w_i - \widetilde{R}u\|_{\widetilde{M}} \le -\|w_i\|_{\widetilde{M}} + \|\widetilde{R}u\|_{\widetilde{M}}$, we have

$$\begin{aligned} \|w_i - \widetilde{R}u\|_{\widetilde{K}} - \sigma\|w_i - \widetilde{R}u\|_{\widetilde{M}} &\le \|w_i\|_{\widetilde{K}} + \|\widetilde{R}u\|_{\widetilde{K}} - \sigma\|w_i\|_{\widetilde{M}} + \sigma\|\widetilde{R}u\|_{\widetilde{M}} \\ &\le C\sqrt{\Phi_i(H,h)}\|u\|_\Lambda, \end{aligned} \tag{6.13}$$

where we have also used Lemma 5.9 in the last step.

From Lemmas 6.6, we know that if $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then

$$\begin{aligned} \|w_i - \widetilde{R}u\|_{\widetilde{M}} &\le C(1+\sigma^2)H^\gamma C_L(H,h)(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_\Lambda) \\ &\le C(1+\sigma^2)H^\gamma C_L(H,h)\left(\|w_i - \widetilde{R}u\|_{\widetilde{K}} + \|u\|_\Lambda\right) \\ &\quad + C\sigma(1+\sigma^2)H^\gamma C_L(H,h)\|w_i - \widetilde{R}u\|_{\widetilde{M}}. \end{aligned}$$

If $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then we can move the last term on the right side of the above inequality to the left and we have

$$\|w_i - \widetilde{R}u\|_{\widetilde{M}} \le C(1+\sigma^2)H^\gamma C_L(H,h)\left(\|w_i - \widetilde{R}u\|_{\widetilde{K}} + \|u\|_\Lambda\right). \tag{6.14}$$

Combining (6.13) and (6.14), we have

$$\|w_i - \widetilde{R}u\|_{\widetilde{K}} \leq C\sigma(1+\sigma^2)H^\gamma C_L(H,h)\left(\|w_i - \widetilde{R}u\|_{\widetilde{K}} + \|u\|_\Lambda\right) + C\sqrt{\Phi_i(H,h)}\|u\|_\Lambda.$$

If $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is small enough, then we have

$$\|w_i - \widetilde{R}u\|_{\widetilde{K}} \leq C(1+\sqrt{\Phi_i(H,h)})\|u\|_\Lambda.$$

□

LEMMA 6.8. *Let Assumption 5.3 hold. Given $u \in \widehat{W}$, let $w_1 = \widetilde{A}^{-1}\widetilde{R}_D A u$, and $w_2 = \widetilde{A}^{-1}(\widetilde{R}_D - J_D^T \mathcal{H}^T)Au$. If $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then*

$$\sigma\|w_i\|_{\widetilde{M}} \leq C(1+\sqrt{\Phi_i(H,h)})\|u\|_\Lambda, \quad i=1,2,$$

where $\Phi_i(H,h)$ are determined as in Lemma 5.5 and C is a positive constant independent of σ, H, and h.

Proof. We know as in the proof of Lemma 6.7 that

$$\left|\ \|w_i\|^2_{\widetilde{K}} - \sigma^2\|w_i\|^2_{\widetilde{M}}\ \right| \leq \|u\|_\Lambda \|E_{D,i}w_i\|_{\widetilde{\Lambda}}.$$

Using Lemma 5.6, we have that if σH is less than a certain constant, then

$$\sigma^2\|w_i\|^2_{\widetilde{M}} - \|w_i\|^2_{\widetilde{K}} \leq C\sqrt{\Phi_i(H,h)}\|u\|_\Lambda\|w_i\|_{\widetilde{\Lambda}}.$$

Dividing by $\sigma\|w_i\|_{\widetilde{M}} + \|w_i\|_{\widetilde{K}}$ on both sides and using Lemma 5.9, we have,

$$\sigma\|w_i\|_{\widetilde{M}} - \|w_i\|_{\widetilde{K}} \leq C\sqrt{\Phi_i(H,h)}\|u\|_\Lambda.$$

Then using the result of Lemma 6.7, we have that if $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then

$$\sigma\|w_i\|_{\widetilde{M}} \leq \|w_i\|_{\widetilde{K}} + C\sqrt{\Phi_i(H,h)}\|u\|_\Lambda \leq C(1+\sqrt{\Phi_i(H,h)})\|u\|_\Lambda.$$

□

In order to confirm Assumption 5.7, we also need the following lemma.

LEMMA 6.9. *Given $u \in \widehat{W}$, let $z_1 = \widetilde{M}^{-1}\widetilde{R}_D M u$ and $z_2 = \widetilde{M}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)Mu$. Then,*

$$\|z_i - u\|_{L^2(\Omega)} \leq CH\sqrt{\Phi_i(H,h)}\ |u|_{H^1(\Omega)}, \quad i=1,2,$$

where $\Phi_i(H,h)$ are determined as in Lemma 5.5, and C is a positive constant independent of σ, H, and h.

Proof. We only give the proof for z_2 in the following; essentially the same argument also applies to z_1. We have

$$\begin{aligned}\|z_2 - u\|_{L^2(\Omega)} &= \|\widetilde{M}^{-1}(\widetilde{R}_D - J_D^T\mathcal{H}^T)\widetilde{R}^T\widetilde{M}\widetilde{R}u - \widetilde{R}u\|_{L^2(\Omega)} \\ &= \|\widetilde{M}^{-1}\left(E_{D,2}^T - I\right)\widetilde{M}\widetilde{R}u\|_{L^2(\Omega)} = \|\widetilde{M}^{-1}P_{D,2}^T\widetilde{M}\widetilde{R}u\|_{L^2(\Omega)} \\ &\leq Ch^{-d}\|P_{D,2}^T\widetilde{M}\widetilde{R}u\|_{L^2(\Omega)},\end{aligned}$$

where we have used Lemma 5.8 in the last step, and that $P_{D,2}^T = I - E_{D,2}^T = I - (\widetilde{R}_D - J_D^T\mathcal{H}^T)\widetilde{R}^T$. Since $P_{D,2}^T\widetilde{M}\widetilde{R}u$ has a zero average over each subdomain interface edge, then by using the Poincaré-Friedrichs inequality, and a result similar to Lemma 5.5, cf. [34, 38, 48, Lemma 4.26], we have

$$\|z_2 - u\|_{L^2(\Omega)} \leq C\frac{H}{h^d}|P_{D,2}^T\widetilde{M}\widetilde{R}u|_{H^1(\Omega)} \leq C\frac{H}{h^d}\sqrt{\Phi_2(H,h)}\ |\widetilde{M}\widetilde{R}u|_{H^1(\Omega)}.$$

Then using Lemma 5.8 again, we have $\|z_2 - u\|_{L^2(\Omega)} \leq CH\sqrt{\Phi_2(H,h)}\ |\widetilde{R}u|_{H^1(\Omega)}$.
□

Using Lemmas 6.6, 6.7, 6.8, and 6.9, we can establish Assumption 5.7.

LEMMA 6.10. *Let Assumption 5.3 hold. Then Assumption 5.7 also holds.*

Proof. Lemmas 6.7 and 6.8 prove the last inequality in Assumption 5.7.

To prove the first inequality in Assumption 5.7, we have, by using Lemma 6.6, that if $\sigma(1+\sigma^2)H^\gamma C_L(H,h)$ is sufficiently small, then

$$\left|\left\langle w_i - \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{M}}\right| \leq \|w_i - \widetilde{R}u\|_{\widetilde{M}}\|u\|_M \leq C(1+\sigma^2)H^\gamma C_L(H,h)(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_\Lambda)\|u\|_M.$$

We know from Lemmas 6.7 and 6.8, that $\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} \leq C(1+\sqrt{\Phi_i(H,h)})\|u\|_\Lambda$. Furthermore since $\|u\|_M \leq \frac{1}{\sigma}\|u\|_\Lambda$, we have

$$\left|\left\langle w_i - \widetilde{R}u, \widetilde{R}u\right\rangle_{\widetilde{M}}\right| \leq C\frac{(1+\sigma^2)}{\sigma}H^\gamma C_L(H,h)(1+\sqrt{\Phi_i(H,h)})\|u\|_\Lambda^2,$$

which proves the first inequality in Assumption 5.7. Similarly, to prove the second inequality in Assumption 5.7, we have, from Lemmas 6.6, 6.7, and 6.8, that

$$\begin{aligned}\left|\left\langle w_i - \widetilde{R}u, w_i\right\rangle_{\widetilde{M}}\right| &\leq C(1+\sigma^2)H^\gamma C_L(H,h)(\|w_i - \widetilde{R}u\|_{\widetilde{\Lambda}} + \|u\|_\Lambda)\|w_i\|_{\widetilde{M}} \\ &\leq C\frac{(1+\sigma^2)}{\sigma}H^\gamma C_L(H,h)\,(1+\Phi_i(H,h))\,\|u\|_\Lambda^2.\end{aligned}$$

To prove the third inequality, we have, from Lemmas 6.9 and 6.8, that

$$\left|\left\langle z_i - \widetilde{R}u, w_i\right\rangle_{\widetilde{M}}\right| \leq C\|z_i - \widetilde{R}u\|_{\widetilde{M}}\|w_i\|_{\widetilde{M}} \leq C\frac{1}{\sigma}H\,(1+\Phi_i(H,h))\,\|u\|_\Lambda^2.$$

□

7. Numerical experiments. FETI-DP methods have been proven successful and parallel scalable for solving a large class of indefinite problems, and their applications include structural dynamics problems, acoustic scattering problems, etc., cf. [19, 16, 20]. Here we use the solution of the problem (2.1) to demonstrate the algorithmic scalability of the proposed BDDC algorithms in this paper, and also demonstrate their spectral equivalence with the FETI-DP methods.

The problem (2.1) is solved on a 2π by 2π square domain with Dirichlet boundary conditions $u = 1$ on the four sides of the square and with $f = 0$. Q1 finite elements are used and the original square domain is decomposed uniformly into square subdomains. In the GMRES iteration, the $\langle\cdot,\cdot\rangle_{K+\sigma^2 M}$ inner product is used; using L_2 inner product gives the same convergence rates. The iteration is stopped when the residual is reduced by 10^{-6}. To have an idea how the shift σ^2 in (2.4) affects the eigenvalues of the matrix $K - \sigma^2 M$, we give the number of negative eigenvalues of $K - \sigma^2 M$ in Table 7.1, for different meshes with 1089, 10201, and 20449 degrees of freedom, respectively, and for different shifts $\sigma^2 = 100$, $\sigma^2 = 200$, and $\sigma^2 = 400$.

TABLE 7.1
Number of negative eigenvalues of $K - \sigma^2 M$ for different meshes and different σ^2.

# of degrees of freedom	$\sigma^2 = 100$	$\sigma^2 = 200$	$\sigma^2 = 400$
1089	243	445	843
10201	290	575	1109
20449	290	585	1161

In our experiments, we test three different choices of the coarse level primal space in our BDDC algorithm. In our first test, the coarse level primal variables are only those at the subdomain corners and no plane wave continuity constraints are enforced across the subdomain edges. This choice does not satisfy Assumption 5.3. In our second test, in addition to the subdomain corner variables, we also include one edge average degree of freedom for each subdomain edge, as required in Assumption 5.3, in the coarse level primal variable space. This edge average degree of freedom corresponds to the vector determined by the cosine plane wave in (4.1) with the angle θ chosen perpendicular to the edge. In our last test, we further add to the coarse level primal space another plane wave continuity constraint on each edge corresponding to the cosine plane wave in (4.1) with the angle θ chosen tangential to the edge. In the following tables, we represent these three different choices of the coarse level primal space by 0-pwa, 1-pwa, and 2-pwa, respectively.

Tables 7.2 and 7.3 show the GMRES iteration counts for the preconditioned operator $B_2^{-1}A$, corresponding to different number of subdomains, different subdomain problem sizes, and the three different choices of the coarse level primal space. With only subdomain corner variables in the coarse level primal space, the convergence cannot be achieved within 300 iterations in most cases. With the inclusion of the edge plane wave augmentations in the coarse level primal space, we see from Table 7.2 that the iteration counts decrease with the increase of the number of subdomains for a fixed subdomain problem size. We see from Table 7.3 that when the number of subdomains is fixed and H/h increases, the iteration counts increase slowly, seemingly in a logarithmic pattern of H/h. Tables 7.2 and 7.3 also show that the convergence becomes slower with the increase of the shift σ^2 and that the convergence rate is improved by including more plane wave continuity constraints in the coarse level primal subspace.

In Table 7.4, we compare the GMRES iteration counts of the BDDC operators $B_1^{-1}A$, $B_2^{-1}A$, and $B_3^{-1}A$ with those of the FETI-DP operators with the lumped preconditioner $B_{D,\Gamma}\widetilde{A}_{\Gamma\Gamma}B_{D,\Gamma}^T$, with the Dirichlet preconditioner $B_{D,\Gamma}\widetilde{S}_{\Gamma}^{\mathcal{H}}B_{D,\Gamma}^T$, and with the Dirichlet preconditioner $B_{D,\Gamma}\widetilde{S}_{\Gamma}B_{D,\Gamma}^T$, respectively. We see that the corresponding BDDC and FETI-DP algorithms have similar convergence rates. We also see that using either subdomain discrete harmonic extension or the extension based on the shifted operator in the BDDC and FETI-DP algorithms gives almost the same convergence rates.

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Table 7.2
Iteration counts for $B_2^{-1}A$ for $H/h = 8$ and changing number of subdomains.

σ^2	# subdomains	Iteration Count		
		0-pwa	1-pwa	2-pwa
100	16×16	183	37	14
	24×24	205	20	7
	32×32	> 300	13	6
200	16×16	> 300	143	112
	24×24	> 300	85	39
	32×32	> 300	47	28
400	16×16	> 300	> 300	236
	24×24	> 300	> 300	75
	32×32	> 300	192	49

Table 7.3
Iteration counts for $B_2^{-1}A$ for 24×24 subdomains and changing H/h.

σ^2	H/h	Iteration Count		
		0-pwa	1-pwa	2-pwa
100	8	205	20	7
	12	188	25	8
	16	182	27	8
200	8	> 300	85	39
	12	> 300	108	60
	16	> 300	114	68
400	8	> 300	> 300	75
	12	> 300	> 300	108
	16	> 300	> 300	111

Table 7.4
Iteration counts for BDDC operators $B_1^{-1}A$, $B_2^{-1}A$, and $B_3^{-1}A$, and for FETI-DP with lumped preconditioner $B_{D,\Gamma}\widetilde{A}_{\Gamma\Gamma}B_{D,\Gamma}^T$, with Dirichlet preconditioner $B_{D,\Gamma}\widetilde{S}_{\Gamma}^{\mathcal{H}}B_{D,\Gamma}^T$, and with Dirichlet preconditioner $B_{D,\Gamma}\widetilde{S}_{\Gamma}B_{D,\Gamma}^T$, for the case $\sigma^2 = 200$ and 2-pwa.

# Subs	H/h	$B_1^{-1}A$	DPH ($\widetilde{A}_{\Gamma\Gamma}$)	$B_2^{-1}A$	DPH ($\widetilde{S}_{\Gamma}^{\mathcal{H}}$)	$B_3^{-1}A$	DPH ($\widetilde{S}_{\Gamma}$)
16×16	8	114	97	112	108	115	106
24×24		40	40	39	39	39	39
32×32		29	30	28	28	28	28
24×24	8	40	40	39	39	39	39
	12	55	54	60	57	58	56
	16	70	70	68	68	67	66

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