INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2000; **00**:1–6 *Prepared using* nmeauth.cls *[Version: 2002/09/18 v2.02]*

A unified FETI-DP approach for incompressible Stokes equations

Xuemin Tu[1,*] and Jing Li[2]

[1] *Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd, Lawrence, KS 66045-7594, U.S.A.*
[2] *Department of Mathematical Sciences, Kent State University, Kent, OH 44242, U.S.A.*

SUMMARY

A unified framework of FETI-DP algorithms is proposed for solving the system of linear equations arising from the mixed finite element approximation of incompressible Stokes equations. A distinctive feature of this framework is that it allows using both continuous and discontinuous pressures in the algorithm, while previous FETI-DP methods only apply to discontinuous pressures. A preconditioned conjugate gradient method is used in the algorithm with either a lumped or a Dirichlet preconditioner, and scalable convergence rates are proved. This framework is also used to describe several previously developed FETI-DP algorithms and greatly simplifies their analysis. Numerical experiments of solving a two-dimensional incompressible Stokes problem demonstrate the performances of the discussed FETI-DP algorithms represented under the same framework.

KEY WORDS: domain decomposition, incompressible Stokes, FETI-DP, BDDC

1. Introduction

The finite element tearing and interconnecting (FETI) methods were introduced by Farhat and Roux [9, 10, 11, 12] for second order elliptic problems. In these algorithms, Lagrange multipliers are introduced to enforce the continuity of the solution across the subdomain interface. The original system of linear equations is reduced to a symmetric positive semi-definite system for the Lagrange multipliers, which can be solved by a preconditioned conjugate gradient method. Both a lumped preconditioner [10] and a Dirichlet preconditioner [12] have been used in the FETI methods. Compared with the lumped preconditioner, the Dirichlet preconditioner is more effective in the reduction of iteration count, but it is also more expensive. Numerical experiments in [12] show that the lumped preconditioner is more efficient for second-order problems while the Dirichlet preconditioner is better for plate and shell problems.

The dual-primal FETI (FETI-DP) method, introduced by Farhat *et. al.* [7, 8], represents a further development of the FETI methods. In a FETI-DP algorithm, a few variables from each

*Correspondence to: Xuemin Tu, Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd, Lawrence, KS 66045-7594, U.S.A.; E-mail: xtu@math.ku.edu

Contract/grant sponsor: National Science Foundation; contract/grant number: DMS-1115759

Received xx/xx/xxxx
Revised xx/xx/xxxx

subdomain are selected as the coarse level primal variables which are shared by neighboring subdomains, while the continuity of the other subdomain interface variables is enforced by using Lagrange multipliers. The reduced system for the Lagrange multipliers becomes symmetric positive definite and is solved by a preconditioned conjugate gradient method using either the lumped or the Dirichlet preconditioner. With an appropriate choice of coarse level variables, the condition number bound of the FETI-DP algorithm has been proved independent of the number of subdomains for both second-order and fourth-order elliptic systems, and in both two and three dimensions, cf. [26, 20].

The FETI-DP algorithm was first extended to solving incompressible Stokes equations by Li [21]. In addition to the coarse level primal velocity variables, the subdomain average pressure degrees of freedom are also selected as the coarse level variables and the resulting coarse level problem is symmetric indefinite. The reduced system for the Lagrange multipliers is still symmetric positive definite and a preconditioned conjugate gradient method can be used. Only the Dirichlet preconditioner was studied in [21], and it was proved for two-dimensional problems that, under the condition that both the subdomain corner and certain edge-average velocity degrees of freedom are selected as coarse level primal variables, the condition number bound is independent of the number of subdomains and grows only polylogarithmically with the size of individual subdomain problems.

Recently, Kim, Lee, and Park [15] introduced a different FETI-DP formulation for solving the incompressible Stokes problems, where no pressure variables are selected as coarse level primal variables and the resulting coarse level problem is symmetric positive definite. Only the lumped preconditioner was studied in [15], for which the edge-average velocity degrees of freedom are no longer needed in the coarse level problem; in fact, as few coarse level primal variables as for solving positive definite elliptic problems were used and as strong condition number bound was established. Reduction of the coarse level problem size has also been achieved by Dohrmann and Widlund in an overlapping Schwarz type algorithm for solving almost incompressible elasticity, cf. [5, 6]. Keeping the size of the coarse problem small is important in large scale computations; a large coarse problem can be a bottleneck and additional efforts in the algorithm are needed to reduce its impact, cf. [32, 33, 31, 18, 4, 16, 34].

All above mentioned algorithms and their analysis for solving incompressible Stokes problems are only valid for finite element discretization using discontinuous pressures. Discontinuous pressures have also been used in domain decomposition algorithms for other similar type saddle-point problems; see [17, 27, 21, 13, 3, 24, 14, 29, 30, 28].

In [22], the authors proposed a new FETI-DP algorithm for solving incompressible Stokes equations, which allows using both discontinuous and continuous pressures in the finite element discretization. The lumped preconditioner was studied in [22] and, similar to [15], as few coarse level primal variables as for solving positive definite elliptic problems were used and as strong condition number bound was established.

The purpose of this paper is two-fold. First, the FETI-DP formulation proposed in [22] is used as a unified framework to describe the two previous FETI-DP algorithms studied in [21] and [15]. It is observed that these two FETI-DP algorithms can be represented as special cases of using discontinuous pressures in the new formulation. The condition number bound estimate based on the new formulation also greatly simplifies the analysis in [21] and [15]. Second, a new Dirichlet preconditioner is studied for the FETI-DP algorithm presented under the unified framework using either continuous or discontinuous pressures. The same condition number bound as in [21] is obtained. Moreover, this new Dirichlet preconditioner involves solving symmetric positive definite subdomain problems and is less expensive compared with the Dirichlet preconditioner used in [21] where subdomain saddle-point problems need be solved. To stay focused on the purpose of this paper, the presentation of the algorithms and their analysis is restricted to the case of solving two-dimensional problems.

The rest of this paper is organized as follows. The finite element discretization of the incompressible Stokes equation is introduced in Section 2. A domain decomposition approach is described in Section 3. A reduced system of equations is derived in Section 4. Section 5 provides some techniques used in the condition number bound estimate. The lumped and the Dirichlet preconditioners are studied in Sections 6 and 7, respectively. At the end, in Section 8, numerical results for solving a two-dimensional incompressible Stokes problem demonstrate the performances of the discussed algorithms and their connections.

2. Finite element discretization

We consider the following incompressible Stokes problem on a bounded, two-dimensional polygonal domain Ω with a Dirichlet boundary condition,

$$\left\{\begin{array}{rcll} -\Delta \mathbf{u} + \nabla p &=& \mathbf{f}, & \text{in } \Omega\ , \\ -\nabla\cdot\mathbf{u} &=& 0, & \text{in } \Omega\ , \\ \mathbf{u} &=& \mathbf{u}_{\partial\Omega}, & \text{on } \partial\Omega\ . \end{array}\right. \tag{1}$$

The boundary velocity $\mathbf{u}_{\partial\Omega}$ satisfies the compatibility condition $\int_{\partial\Omega}\mathbf{u}_{\partial\Omega}\cdot\mathbf{n}=0$. Without loss of generality, we assume that $\mathbf{u}_{\partial\Omega}=\mathbf{0}$.

The weak solution of (1) is given by: find $\mathbf{u}\in\left(H_0^1(\Omega)\right)^2=\{\mathbf{v}\in(H^1(\Omega))^2 \mid \mathbf{v}=\mathbf{0} \text{ on } \partial\Omega\}$, and $p\in L^2(\Omega)$, such that,

$$\left\{\begin{array}{lcll} a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p) &=& (\mathbf{f},\mathbf{v}), & \forall \mathbf{v}\in\left(H_0^1(\Omega)\right)^2\ , \\ b(\mathbf{u},q) &=& 0, & \forall q\in L^2(\Omega)\ , \end{array}\right. \tag{2}$$

where

$$a(\mathbf{u},\mathbf{v})=\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v},\quad b(\mathbf{u},q)=-\int_\Omega(\nabla\cdot\mathbf{u})q,\quad (\mathbf{f},\mathbf{v})=\int_\Omega \mathbf{f}\cdot\mathbf{v}.$$

The solution of (2) is not unique, with p different up to an additive constant.

A mixed finite element method is used to solve (2). Ω is triangulated into shape-regular elements of characteristic size h. $\mathbf{W}\in\left(H_0^1(\Omega)\right)^2$ represents the velocity finite element space and it contains continuous functions. The pressure finite element space is represented by $Q\subset L^2(\Omega)$. Both continuous and discontinuous pressures can be used in our algorithm. A mixed finite element space with discontinuous pressure is shown on the left in Figure 1 on a uniform triangular mesh of a rectangular domain, where the velocity is piecewise linear on the mesh and the pressure is a constant on each union of four triangles as shown on the right in the figure. This mixed finite element was used in [21].

A mixed finite element space with continuous pressure is the modified Taylor-Hood mixed finite element, as shown in Figure 2, cf. [1, Chapter III, §7]. The velocity finite element space contains the piecewise linear functions on the finest triangular mesh and the pressure finite element space contains the piecewise linear functions on the coarser triangular mesh with the doubled mesh size.

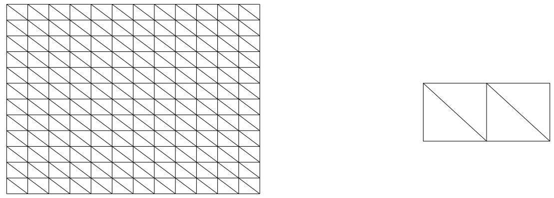

Figure 1. A mixed finite element with discontinuous pressures.

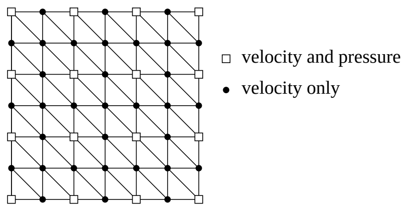

Figure 2. The modified Taylor-Hood mixed finite element.

The finite element solution $(\mathbf{u}, p) \in \mathbf{W} \bigoplus Q$ of (2) satisfies

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}. \tag{3}$$

Here A, B and $\mathbf{f}$ represent respectively the restrictions of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $\mathbf{f}(\cdot)$ to the finite-dimensional spaces $\mathbf{W}$ and Q. In this paper the same notation is used to represent both a finite element function and the vector of its nodal values.

The coefficient matrix in (3) is rank deficient. A is symmetric positive definite. The kernel of B^T, denoted by $Ker(B^T)$, is the space of all constant pressures in Q. The range of B, denoted by $Im(B)$, is orthogonal to $Ker(B^T)$ and is the subspace of Q consisting of all vectors with zero average. The solution of (3) always exists and is uniquely determined when the pressure is considered in the quotient space $Q/Ker(B^T)$. In this paper, when $q \in Q/Ker(B^T)$, q always has zero average. For a more general right-hand side vector $(\mathbf{f},\ g)$ given in (3), the existence of its solution requires that $g \in Im(B)$, i.e., g has zero average.

Both mixed finite elements shown on Figures 1 and 2 are inf-sup stable in the sense that there exists a positive constant β, independent of h, such that

$$\sup_{\mathbf{w}\in\mathbf{W}} \frac{\langle q, B\mathbf{w}\rangle^2}{\langle \mathbf{w}, A\mathbf{w}\rangle} \geq \beta^2 \langle q, Zq\rangle, \qquad \forall q \in Q/Ker(B^T). \tag{4}$$

Here, as always in this paper, $\langle\cdot,\cdot\rangle$ represents the inner product of two vectors. The matrix Z represents the mass matrix defined on the pressure finite element space, i.e., for any $q \in Q$, $\|q\|_{L^2}^2 = \langle q, Zq\rangle$. Z is spectrally equivalent to $h^2 I$, where I represents the identity matrix of the same dimension, i.e., there exist positive constants c and C, such that

$$ch^2 I \leq Z \leq Ch^2 I, \tag{5}$$

cf. [35, Lemma B.31]. Here, and in other places of this paper, c and C represent generic positive constants which are independent of the mesh size h and H (discussed in the following section).

3. A non-overlapping domain decomposition approach

The domain Ω is decomposed into N nonoverlapping polygonal subdomains Ω_i, $i = 1, 2, ..., N$. Each subdomain is a union of a bounded number of elements, with the diameter of each subdomain in the order of H. The nodes on the boundaries of neighboring subdomains match across the subdomain interface $\Gamma = (\cup \partial\Omega_i)\backslash\partial\Omega$. Γ is composed of subdomain edges, which are regarded as open subsets of Γ, and of the subdomain vertices, which are end points of edges.

The velocity and pressure finite element spaces $\mathbf{W}$ and Q are decomposed into

$$\mathbf{W} = \mathbf{W}_I \bigoplus \mathbf{W}_\Gamma, \quad Q = Q_I \bigoplus Q_\Gamma,$$

respectively. Here $\mathbf{W}_I$ and Q_I are the direct sums of subdomain interior velocity spaces $\mathbf{W}_I^{(i)}$, and subdomain interior pressure spaces $Q_I^{(i)}$, respectively, i.e.,

$$\mathbf{W}_I = \bigoplus_{i=1}^{N} \mathbf{W}_I^{(i)}, \quad Q_I = \bigoplus_{i=1}^{N} Q_I^{(i)}.$$

$\mathbf{W}_\Gamma$ is the subdomain boundary velocity space and it contains the subdomain boundary velocity degrees of freedom shared by neighboring subdomains. Q_Γ contains the subdomain boundary pressure degrees of freedom shared by neighboring subdomains. For the case of using discontinuous pressures, no pressure degrees of freedom are shared by neighboring subdomains and Q_Γ is empty. In fact, for the discontinuous pressure case, the algorithm presented in this paper allows that Q_Γ either be empty or contain any number of intrinsically subdomain interior pressure degrees of freedom; more details on this will be discussed in Sections 4.1 and 4.2.

To formulate our domain decomposition algorithm, we introduce a partially sub-assembled interface velocity space

$$\widetilde{\mathbf{W}}_\Gamma = \mathbf{W}_\Pi \bigoplus \mathbf{W}_\Delta = \mathbf{W}_\Pi \bigoplus \left(\bigoplus_{i=1}^{N} \mathbf{W}_\Delta^{(i)} \right).$$

Here, $\mathbf{W}_\Pi$ is the continuous coarse level velocity space and the coarse level primal velocity degrees of freedom are shared by neighboring subdomains. The complimentary space $\mathbf{W}_\Delta$ is the direct sum of subdomain dual interface velocity spaces $\mathbf{W}_\Delta^{(i)}$, which correspond to the remaining interface velocity degrees of freedom and are spanned by basis functions which vanish at the primal degrees of freedom. Thus, an element in the space $\widetilde{\mathbf{W}}_\Gamma$ has a continuous primal velocity component and typically a discontinuous dual velocity component.

In this paper, two choices of $\mathbf{W}_\Pi$ are used. In the first, $\mathbf{W}_\Pi$ is spanned by all the subdomain corner velocity nodal basis functions and the coarse level primal variables are only the subdomain corner velocity variables. In the second, besides all the subdomain corner velocity nodal basis functions, on each edge Γ^{ij} shared by neighboring subdomains Ω_i and Ω_j, $\mathbf{W}_\Pi$ is also spanned by an edge-average finite element basis function such that $\int_{\Gamma^{ij}} \mathbf{w}_\Delta^{(i)} \cdot \mathbf{n}_{ij} = 0$, for any $\mathbf{w}_\Delta^{(i)} \in \mathbf{W}_\Delta^{(i)}$. Here $\mathbf{n}_{ij}$ denotes a fixed selection of the normal to Γ^{ij}. Therefore for the second choice of $\mathbf{W}_\Pi$, the following divergence free boundary condition

$$\int_{\partial\Omega_i} \mathbf{w}_\Delta^{(i)} \cdot \mathbf{n} = 0 \tag{6}$$

is satisfied for all $\mathbf{w}_{\Delta}^{(i)} \in \mathbf{W}_{\Delta}^{(i)}$. For more details on choosing coarse level primal edge-average velocity variables to satisfy the divergence free condition for incompressible Stokes problems, including for the three-dimensional case, see [24, Section 7]. We note that the choice of $\mathbf{W}_{\Pi}$ depends on the preconditioner used in the algorithm. The first choice is sufficient for using the lumped preconditioner, but for the Dirichlet preconditioner the second one has to be used; for more detailed discussions, see Sections 6 and 7.

The functions $\mathbf{w}_{\Delta}$ in $\mathbf{W}_{\Delta}$ are in general not continuous across Γ. To enforce their continuity, a boolean matrix B_{Δ} is constructed from $\{0, 1, -1\}$. Each row of B_{Δ} only contains two non-zero entries, 1 and -1, corresponding to the same velocity degree of freedom on each subdomain boundary node, but attributed to two neighboring subdomains. For any $\mathbf{w}_{\Delta}$ in $\mathbf{W}_{\Delta}$, each row of $B_{\Delta}\mathbf{w}_{\Delta} = 0$ implies that the two degrees of freedom from the two neighboring subdomains be the same. When non-redundant continuity constraints are enforced, B_{Δ} has full row rank. We denote the range of B_{Δ} applied on $\mathbf{W}_{\Delta}$ by Λ, the vector space of the Lagrange multipliers.

In order to define a certain subdomain boundary scaling operator, we introduce a positive scaling factor $\delta^{\dagger}(x)$ for each node x on the subdomain boundary Γ. Let $\mathcal{N}_x$ be the number of subdomains sharing x, and we simply take $\delta^{\dagger}(x) = 1/\mathcal{N}_x$. In applications, these scaling factors will depend on the heat conduction coefficient and the first of the Lamé parameters for scalar elliptic problems and the equations of linear elasticity, respectively; see [20, 19]. Given such scaling factors on the subdomain boundary nodes, we can define a scaled operator $B_{\Delta,D}$. We recall that each row of B_{Δ} has only two nonzero entries, 1 and -1, corresponding to the same subdomain boundary node x. Multiplying each entry by the scaling factor $\delta^{\dagger}(x)$ gives us $B_{\Delta,D}$.

Then solving the original fully assembled linear system (3) is equivalent to: find $(\mathbf{u}_I,\ p_I,\ \mathbf{u}_{\Delta},\ \mathbf{u}_{\Pi},\ p_{\Gamma},\ \lambda) \in \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_{\Delta} \bigoplus \mathbf{W}_{\Pi} \bigoplus Q_{\Gamma} \bigoplus \Lambda$, such that

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} & B_{\Gamma I}^T & 0 \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} & 0 & 0 \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} & B_{\Gamma\Delta}^T & B_{\Delta}^T \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} & A_{\Pi\Pi} & B_{\Gamma\Pi}^T & 0 \\ B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} & 0 & 0 \\ 0 & 0 & B_{\Delta} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_{\Delta} \\ \mathbf{u}_{\Pi} \\ p_{\Gamma} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_{\Delta} \\ \mathbf{f}_{\Pi} \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

Corresponding to the one-dimensional null space of (3), we consider a vector of the form $(\mathbf{u}_I,\ p_I,\ \mathbf{u}_{\Delta},\ \mathbf{u}_{\Pi},\ p_{\Gamma},\ \lambda) = (\mathbf{0},\ 1_{p_I},\ \mathbf{0},\ \mathbf{0},\ 1_{p_{\Gamma}}, \lambda)$, where 1_{p_I} and $1_{p_{\Gamma}}$ represent vectors with value 1 on each entry of the vector. Substituting it into (7) gives zero blocks on the right-hand side, except at the third block

$$\mathbf{f}_{\Delta} = [B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_{\Gamma}} \end{bmatrix} + B_{\Delta}^T \lambda. \tag{8}$$

The first term in (8) represents the line integral of the normal components of the velocity finite element basis functions across the subdomain boundary, and corresponding to the same velocity degree of freedom on the subdomain boundary, their values on the two neighboring subdomains are negative of each other. Therefore

$$[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_{\Gamma}} \end{bmatrix} = B_{\Delta}^T B_{\Delta,D} [B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_{\Gamma}} \end{bmatrix},$$

from which we know that $\mathbf{f}_\Delta = \mathbf{0}$ in (8), for

$$\lambda = -B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix}.$$

Therefore, a basis of the one-dimensional null space of (7) is

$$\left(0, \quad 1_{p_I}, \quad 0, \quad 0, \quad 1_{p_\Gamma}, \quad -B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} \right), \tag{9}$$

which is also valid if p_Γ in (7) is empty and then the block 1_{p_Γ} in (9) disappears.

For the case when the coarse level primal velocity space $\mathbf{W}_\Pi$ contains both the subdomain corner and edge-average variables such that the divergence free boundary condition (6) is satisfied, we have $\int_{\Omega_i} \nabla \cdot \mathbf{w}_\Delta^{(i)} = 0$, for all $\mathbf{w}_\Delta^{(i)} \in \mathbf{W}_\Delta^{(i)}$, which is in matrix form $\left(1_{p_I^{(i)}}^T B_{I\Delta}^{(i)} + 1_{p_\Gamma^{(i)}}^T B_{\Gamma\Delta}^{(i)}\right) \mathbf{w}_\Delta^{(i)} = 0$. As a result

$$[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} = 0. \tag{10}$$

Therefore, when the divergence free boundary condition (6) is enforced and p_Γ in (7) is empty, we know from (10) and (9) that the leading four-by-four diagonal block in the coefficient matrix of (7) is singular and its null space consists of all vectors with a constant pressure and zero velocity.

4. Reduced system of linear equations

We first describe the FETI-DP formulation proposed in [22] and then use it as a framework to represent the two previous FETI-DP algorithms studied in [21] and [15]. Based on (7), denote

$$\widetilde{A} = \begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix}, \quad B_C = \begin{bmatrix} B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} \\ 0 & 0 & B_\Delta & 0 \end{bmatrix}, \quad f = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \end{bmatrix}. \tag{11}$$

The variables $(\mathbf{u}_I,\ p_I,\ \mathbf{u}_\Delta,\ \mathbf{u}_\Pi)$ can be eliminated from (7) and we obtain a Schur complement problem for the variables $(p_\Gamma,\ \lambda)$

$$G \begin{bmatrix} p_\Gamma \\ \lambda \end{bmatrix} = g, \tag{12}$$

where

$$G = B_C \widetilde{A}^{-1} B_C^T, \quad g = B_C \widetilde{A}^{-1} f. \tag{13}$$

Remark 1. *The only case that $\widetilde{A}$ is singular in the algorithm is when p_Γ in (7) is empty and the divergence free boundary condition (6) is enforced, as discussed at the end of Section 3. However the definitions in (13) are still valid, since f and columns of B_C^T are in the range of $\widetilde{A}$ and the kernel of $\widetilde{A}$ is a subspace of the kernel of B_C. For the simplicity of notation, we still use $\widetilde{A}^{-1}$ in (13) and in other places of this paper to represent the solution of system of linear equations, not necessarily the inverse of $\widetilde{A}$.*

We can see that $-G$ is the Schur complement of the coefficient matrix of (7) with respect to the last two row blocks:

$$\begin{bmatrix} I & 0 \\ -B_C\widetilde{A}^{-1} & I \end{bmatrix} \begin{bmatrix} \widetilde{A} & B_C^T \\ B_C & 0 \end{bmatrix} \begin{bmatrix} I & -\widetilde{A}^{-1}B_C^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & -G \end{bmatrix}.$$

If $\widetilde{A}$ is nonsingular, then from the Sylvester law of inertia we can see that G is symmetric positive semi-definite and its null space is derived from the null space of the original coefficient matrix of (7), cf. (9), and a basis is given by,

$$\left(1_{p_\Gamma}, \quad -B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} \right).$$

We denote $X = Q_\Gamma \bigoplus \Lambda$. The range space of G, denoted by R_G, is a subspace of X. R_G is orthogonal to the null space of G and thus has the form

$$R_G = \left\{ \begin{bmatrix} g_{p_\Gamma} \\ g_\lambda \end{bmatrix} \in X : g_{p_\Gamma}^T 1_{p_\Gamma} - g_\lambda^T \left(B_{\Delta,D}[B_{I\Delta}^T \quad B_{\Gamma\Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_\Gamma} \end{bmatrix} \right) = 0 \right\}. \tag{14}$$

When p_Γ in (7) is empty and the divergence free boundary condition (6) is enforced, as discussed in Remark 1, $\widetilde{A}$ is singular and then G becomes positive definite. The range space formula (14) is still valid, cf. (10), and in fact R_G becomes the whole Λ.

In both cases, the restriction of G to its range R_G is positive definite. The fact that the solution of (7) always exists for any given $(\mathbf{f}_I, \mathbf{f}_\Delta, \mathbf{f}_\Pi)$ on the right-hand side implies that the solution of (12) exits for any g defined in (13). Therefore $g \in R_G$. When the conjugate gradient method is applied to solve (12) with zero initial guess, all the iterates are in the Krylov subspace generated by G and g, which is also a subspace of R_G, and where the conjugate gradient method cannot break down. After obtaining (p_Γ, λ) from solving (12), the other components $(\mathbf{u}_I, p_I, \mathbf{u}_\Delta, \mathbf{u}_\Pi)$ in (7) are obtained by back substitution.

The main computation of multiplying G by a vector is the product of $\widetilde{A}^{-1}$ with a vector in the structure of f. We denote

$$A_{rr} = \begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} \\ B_{II} & 0 & B_{I\Delta} \\ A_{\Delta I} & B_{I\Delta}^T & A_{\Delta\Delta} \end{bmatrix}, \quad A_{\Pi r} = A_{r\Pi}^T = \begin{bmatrix} A_{\Pi I} & B_{I\Pi}^T & A_{\Pi\Delta} \end{bmatrix}, \quad f_r = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \end{bmatrix},$$

and define the Schur complement (coarse level problem)

$$S_\Pi = A_{\Pi\Pi} - A_{\Pi r}A_{rr}^{-1}A_{r\Pi}.$$

Then the product $\widetilde{A}^{-1}f$ can be represented by

$$\begin{bmatrix} A_{rr}^{-1}f_r \\ 0 \end{bmatrix} + \begin{bmatrix} -A_{rr}^{-1}A_{r\Pi} \\ I_\Pi \end{bmatrix} S_\Pi^{-1} \left(\mathbf{f}_\Pi - A_{\Pi r}A_{rr}^{-1}f_r\right), \tag{15}$$

which requires solving two subdomain Neumann type problems and one coarse level problem.

The formulation of the reduced system (12) is valid for using both discontinuous and continuous pressures in the algorithm, which is discussed in the following based on whether p_Γ is empty or not. We note that when the discontinuous pressure is used in the algorithm and p_Γ is empty, (12) is the same system as those obtained in [21] and [15].

4.1. p_Γ *is non-empty*

This happens when a continuous pressure finite element space is used, where p_Γ represents all the subdomain boundary pressure degrees of freedom shared by neighboring subdomains. Then (12) is a system for both the subdomain boundary pressures and the Lagrange multipliers. $\widetilde{A}$ in (13) and A_{rr} in (15) are both invertible and the coarse level problem operator S_Π is symmetric positive definite.

p_Γ can also be non-empty for the case of using discontinuous pressures. Since there are no pressure degrees of freedom shared by neighboring subdomains, it is free in the algorithm to choose any number of subdomain pressure variables as p_Γ. If p_Γ contains at least one pressure degree of freedom from each subdomain, then $\widetilde{A}$ in (13) and A_{rr} in (15) are still invertible and S_Π is symmetric positive definite.

4.2. p_Γ *is empty*

p_Γ can be empty only when discontinuous pressures are used in the finite element space since no pressure degrees of freedom are shared by neighboring subdomains.

When p_Γ is empty, (12) becomes a system for the Lagrange multipliers only and it is the same equation as those obtained in [21] and [15]. The implementation of the product of $\widetilde{A}^{-1}$ with a vector as specified in (15) is the same as in [15] and the resulting S_Π is symmetric positive definite. Kim *et. al.* [15] considered only the first choice of the coarse level primal velocity space $\mathbf{W}_\Pi$, namely it contains only the subdomain corner velocities, for which they proved in [15, Lemma 3.1] that $\widetilde{A}$ and A_{rr} are both invertible. However, $\widetilde{A}$ and A_{rr} both become singular when the second choice of $\mathbf{W}_\Pi$ is used to enforce the divergence free boundary condition (6) (required for using the Dirichlet preconditioner, cf. Section 7), even though their singularities do not affect the multiplication of $\widetilde{A}^{-1}$ by a vector; see Remark 1.

In [21], the divergence free boundary condition (6) is enforced for using the Dirichlet preconditioner and a different implementation of multiplying $\widetilde{A}^{-1}$ with a vector was used to avoid the singularity of A_{rr} in (15). There the subdomain constant pressures are pulled out from Q_I to form another vector p_0 and Q_I contains only subdomain interior pressures with zero average on each subdomain; p_0 is combined with the coarse level primal velocity variables to form the coarse level problem. More precisely, $\widetilde{A}$ in (11) is represented by

$$\widetilde{A} = \begin{bmatrix} A_{II} & B_{II}^{-T} & A_{I\Delta} & A_{I\Pi} & B_{II}^{0^T} \\ B_{II}^{-} & 0 & B_{I\Delta}^{-} & B_{I\Pi}^{-} & 0 \\ A_{\Delta I} & B_{I\Delta}^{-T} & A_{\Delta\Delta} & A_{\Delta\Pi} & B_{I\Delta}^{0^T} \\ A_{\Pi I} & B_{I\Pi}^{-T} & A_{\Pi\Delta} & A_{\Pi\Pi} & B_{I\Pi}^{0^T} \\ B_{II}^{0} & 0 & B_{I\Delta}^{0} & B_{I\Pi}^{0} & 0 \end{bmatrix},$$

where, e.g., B_{II}^- and B_{II}^0, represent blocks corresponding to the subdomain interior pressures with zero average and the subdomain constant pressures, respectively. Denote correspondingly

$$A_{rr} = \begin{bmatrix} A_{II} & B_{II}^{-T} & A_{I\Delta} \\ B_{II}^{-} & 0 & B_{I\Delta}^{-} \\ A_{\Delta I} & B_{I\Delta}^{-T} & A_{\Delta\Delta} \end{bmatrix}, \quad A_{\Pi r} = A_{r\Pi}^T = \begin{bmatrix} A_{\Pi I} & B_{I\Pi}^{-T} & A_{\Pi\Delta} \\ B_{II}^{0} & 0 & B_{I\Delta}^{0} \end{bmatrix}, \quad f_r = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \end{bmatrix},$$

Prepared using nmeauth.cls

where A_{rr} is invertible, and define the Schur complement (coarse level problem)

$$S_\Pi = \begin{bmatrix} A_{\Pi\Pi} & B_{I\Pi}^{0^T} \\ B_{I\Pi}^0 & 0 \end{bmatrix} - A_{\Pi r} A_{rr}^{-1} A_{r\Pi},$$

which is a saddle point problem. Then the product $\widetilde{A}^{-1} f$ can be represented by

$$\begin{bmatrix} A_{rr}^{-1} f_r \\ 0 \end{bmatrix} + \begin{bmatrix} -A_{rr}^{-1} A_{r\Pi} \\ I_\Pi \end{bmatrix} S_\Pi^{-1} \left(\begin{bmatrix} \mathbf{f}_\Pi \\ 0 \end{bmatrix} - A_{\Pi r} A_{rr}^{-1} f_r \right).$$

Remark 2. *Even though the implementations of multiplying $\widetilde{A}^{-1}$ with a vector proposed in [21] and [15] are different, the same system (12) for the Lagrange multipliers is solved. When equipped with the same type preconditioners, their convergence rates are the same.*

5. Some techniques

We first define certain norms for several vector/function spaces. We denote

$$\widetilde{\mathbf{W}} = \mathbf{W}_I \bigoplus \widetilde{\mathbf{W}}_\Gamma. \tag{16}$$

For any $\mathbf{w}$ in $\widetilde{\mathbf{W}}$, denote its restriction to subdomain Ω_i by $\mathbf{w}^{(i)}$. A subdomain-wise H^1-seminorm can be defined for functions in $\widetilde{\mathbf{W}}$ by

$$|\mathbf{w}|_{H^1}^2 = \sum_{i=1}^N |\mathbf{w}^{(i)}|_{H^1(\Omega_i)}^2.$$

Several of the following lemmas have been proved in [22]; they are presented here for the completeness of this paper. We denote in (7)

$$\widetilde{B} = \begin{bmatrix} B_{II} & B_{I\Delta} & B_{I\Pi} \\ B_{\Gamma I} & B_{\Gamma\Delta} & B_{\Gamma\Pi} \end{bmatrix}. \tag{17}$$

The following lemma on the stability of $\widetilde{B}$ is [22, Lemma 5.1].

Lemma 1. *For any $\mathbf{w} \in \widetilde{\mathbf{W}}$ and $q \in Q$, $\left\langle \widetilde{B}\mathbf{w}, q \right\rangle \le |\mathbf{w}|_{H^1} \|q\|_{L^2}$.*

Proof:
$$\left\langle \widetilde{B}\mathbf{w}, q \right\rangle^2 = \left(\sum_{i=1}^N \int_{\Omega_i} \nabla \cdot \mathbf{w}^{(i)} q \right)^2 \le \left(\sum_{i=1}^N \sqrt{\int_{\Omega_i} |\nabla \mathbf{w}^{(i)}|^2} \sqrt{\int_{\Omega_i} q^2} \right)^2$$
$$\le \left(\sum_{i=1}^N \int_{\Omega_i} |\nabla \mathbf{w}^{(i)}|^2 \right) \left(\sum_{i=1}^N \int_{\Omega_i} q^2 \right) = |\mathbf{w}|_{H^1}^2 \|q\|_{L^2}^2. \qquad \square$$

We define

$$W = \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi,$$

and its subspace

$$\widetilde{W}_0 = \{w = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in W : B_{II}\mathbf{w}_I + B_{I\Delta}\mathbf{w}_\Delta + B_{I\Pi}\mathbf{w}_\Pi = 0\}. \tag{18}$$

For any $w = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{W}_0$,

$$\langle w, w\rangle_{\widetilde{A}} = \sum_{i=1}^{N} \begin{bmatrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_\Delta^{(i)} \\ \mathbf{w}_\Pi^{(i)} \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & A_{I\Pi}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Delta\Pi}^{(i)} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_\Delta^{(i)} \\ \mathbf{w}_\Pi^{(i)} \end{bmatrix} = \sum_{i=1}^{N} \left\| \begin{bmatrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_\Delta^{(i)} \\ \mathbf{w}_\Pi^{(i)} \end{bmatrix} \right\|_{H^1(\Omega^i)}^2, \tag{19}$$

i.e., $\langle \cdot, \cdot \rangle_{\widetilde{A}}$ defines a seminorm on $\widetilde{W}_0$. In (19), the superscript $^{(i)}$ is used to represent the restrictions of corresponding vectors and matrices to subdomain Ω_i.

The following lemma is [22, Lemma 6.6].

Lemma 2. *For any $w = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{W}_0$, $B_C w \in R_G$.*

Proof: We know for any $(\mathbf{f}_I,\ \mathbf{f}_\Delta,\ \mathbf{f}_\Pi) \in \mathbf{W}_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi$, g defined in (13) is in R_G. For any $w = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{W}_0$, from the definition of $\widetilde{A}$ in (11), there always exists $(\mathbf{f}_I,\ \mathbf{f}_\Delta,\ \mathbf{f}_\Pi) \in \mathbf{W}_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi$, such that

$$\widetilde{A}w = \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \end{bmatrix}, \quad \text{i.e.,} \quad w = \widetilde{A}^{-1} \begin{bmatrix} \mathbf{f}_I \\ 0 \\ \mathbf{f}_\Delta \\ \mathbf{f}_\Pi \end{bmatrix}.$$

Taking such $(\mathbf{f}_I,\ \mathbf{f}_\Delta,\ \mathbf{f}_\Pi)$, g defined in (13) is $B_C w$. □

From (11), we denote the first row of B_C by

$$\widetilde{B}_\Gamma = [B_{\Gamma I} \quad 0 \quad B_{\Gamma\Delta} \quad B_{\Gamma\Pi}];$$

for the second row, we denote the restriction from W onto $\mathbf{W}_\Delta$ by $\widetilde{R}_\Delta$, such that for any $w = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in W$, $\widetilde{R}_\Delta w = \mathbf{w}_\Delta$. Then G, defined in (13), can be written as the following two-by-two block structure

$$G = \begin{bmatrix} G_{p_\Gamma p_\Gamma} & G_{p_\Gamma \lambda} \\ G_{\lambda p_\Gamma} & G_{\lambda\lambda} \end{bmatrix}, \tag{20}$$

where

$$G_{p_\Gamma p_\Gamma} = \widetilde{B}_\Gamma \widetilde{A}^{-1} \widetilde{B}_\Gamma^T, \qquad G_{p_\Gamma \lambda} = \widetilde{B}_\Gamma \widetilde{A}^{-1} \widetilde{R}_\Delta^T B_\Delta^T,$$
$$G_{\lambda p_\Gamma} = B_\Delta \widetilde{R}_\Delta \widetilde{A}^{-1} \widetilde{B}_\Gamma^T, \qquad G_{\lambda\lambda} = B_\Delta \widetilde{R}_\Delta \widetilde{A}^{-1} \widetilde{R}_\Delta^T B_\Delta^T.$$

We define a certain jump operator across the subdomain interface Γ. Let $P_{D,1} : W \to W$, be defined by

$$P_{D,1} = \widetilde{R}_\Delta^T B_{\Delta,D}^T B_\Delta \widetilde{R}_\Delta,$$

cf. [23]. We can see that application of $P_{D,1}$ to a vector essentially computes the difference (jump) of the dual velocity components across the subdomain interface and then distributes the jump to neighboring subdomains according to the scaling factor $\delta^\dagger(x)$. In fact, from the definition of $P_{D,1}$, the only component involved in its application is the component in the space $\mathbf{W}_\Delta$; all other components are kept zero and they are added into the definition to make $P_{D,1}$ more convenient to use in the analysis.

Note that for any $w \in W$, $\langle P_{D,1}w, P_{D,1}w\rangle_{\widetilde{A}} = \left\langle B_{\Delta,D}^T B_\Delta \mathbf{w}_\Delta, B_{\Delta,D}^T B_\Delta \mathbf{w}_\Delta \right\rangle_{A_{\Delta\Delta}}$. The following lemma can be found essentially from [25, Section 6].

Lemma 3. *There exists a function $\Phi_1(H/h)$, such that $\langle P_{D,1}w, P_{D,1}w\rangle_{\widetilde{A}} \leq \Phi_1(H/h)\,\langle w, w\rangle_{\widetilde{A}}$, for all $w \in \widetilde{W}_0$. For two-dimensional problems, when the coarse level primal velocity space contains the subdomain corner velocity variables, $\Phi_1(H/h) = C(H/h)(1+\log(H/h))$.*

To improve the bound on the jump operator, the jumps across the subdomain interface can be extended to the interior of subdomains by subdomain discrete harmonic extension. We define a Schur complement $H_\Delta^{(i)} : \mathbf{W}_\Delta^{(i)} \rightarrow \mathbf{W}_\Delta^{(i)}$ by, for any $\mathbf{w}_\Delta^{(i)} \in \mathbf{W}_\Delta^{(i)}$,

$$\begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_\Delta^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ H_\Delta^{(i)}\mathbf{w}_\Delta^{(i)} \end{bmatrix}. \tag{21}$$

We can see that to multiply $H_\Delta^{(i)}$ by the vector $\mathbf{w}_\Delta^{(i)}$, subdomain elliptic problems with given boundary velocity $\mathbf{w}_\Delta^{(i)}$ and $\mathbf{w}_\Pi^{(i)} = \mathbf{0}$ need be solved. Using $H_\Delta^{(i)}$, we define the second jump operator $P_{D,2} : W \rightarrow W$, by: for any given $w = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in W$, on each subdomain Ω_i, the subdomain interior velocity part of $P_{D,2}w$ is taken as $\mathbf{w}_I^{(i)}$ in the solution of (21), with given subdomain boundary velocity $\mathbf{w}_\Delta^{(i)} = B_{\Delta,D}^{(i)^T} B_\Delta \mathbf{w}_\Delta$. Here $B_{\Delta,D}^{(i)^T}$ represents restriction of $B_{\Delta,D}^T$ on subdomain Ω_i and is a map from Λ to $\mathbf{W}_\Delta^{(i)}$. The other components of $P_{D,2}w$ are kept zero. Therefore

$$\begin{aligned} \langle P_{D,2}w, P_{D,2}w\rangle_{\widetilde{A}} &= \sum_{i=1}^N \begin{bmatrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_\Delta^{(i)} \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_I^{(i)} \\ \mathbf{w}_\Delta^{(i)} \end{bmatrix} \\ &= \sum_{i=1}^N \mathbf{w}_\Delta^{(i)^T} H_\Delta^{(i)} \mathbf{w}_\Delta^{(i)} = \sum_{i=1}^N \mathbf{w}_\Delta^T B_\Delta^T B_{\Delta,D}^{(i)} H_\Delta^{(i)} B_{\Delta,D}^{(i)^T} B_\Delta \mathbf{w}_\Delta \\ &= \sum_{i=1}^N \left| \begin{bmatrix} B_{\Delta,D}^{(i)^T} B_\Delta \mathbf{w}_\Delta \\ 0 \end{bmatrix} \right|^2_{H^{1/2}(\partial\Omega^i)} \leq \Phi_2(H/h) \sum_{i=1}^N |\mathbf{w}_\Gamma^{(i)}|^2_{H^{1/2}(\partial\Omega^i)}, \end{aligned} \tag{22}$$

where $\mathbf{w}_\Gamma^{(i)}$ represents the restriction of $(\mathbf{w}_\Delta,\ \mathbf{w}_\Pi)$ on subdomain Ω_i. The last inequality in (22) is a well established result, cf., [35, Lemma 6.34]. It has been established that, $\Phi_2(H/h) = C(1+\log(H/h))^2$, for two-dimensional problems, when the coarse level primal velocity space contains the subdomain corner velocity variables, cf. [36, Lemma 3.3].

Then from (22) and (19), we have

Lemma 4. *There exists a function $\Phi_2(H/h)$, such that $\langle P_{D,2}w, P_{D,2}w\rangle_{\widetilde{A}} \leq \Phi_2(H/h)\,\langle w, w\rangle_{\widetilde{A}}$, for all $w \in \widetilde{W}_0$. For two-dimensional problems, when the coarse level primal velocity space contains the subdomain corner velocity variables, $\Phi_2(H/h) = C(1+\log(H/h))^2$.*

The following lemma is also used in our analysis and can be found at [13, Lemma 2.3].

Lemma 5. *Consider the saddle point problem: find* $(\mathbf{u}, p) \in \mathbf{W} \bigoplus Q$*, such that*

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \end{bmatrix}, \tag{23}$$

where A *and* B *are as in (3),* $\mathbf{f} \in \mathbf{W}$*,* $g \in Im(B) \subset Q$*. Let* β *be the inf-sup constant specified in (4). Then* $\|\mathbf{u}\|_A \leq \|\mathbf{f}\|_{A^{-1}} + \frac{1}{\beta}\|g\|_{Z^{-1}}$*, where* Z *is the mass matrix defined in Section 2.*

The lumped and the Dirichlet preconditioners for solving (12) will be studied in Sections 6 and 7. Let M^{-1} be symmetric positive definite. When the conjugate gradient method is applied to solve the preconditioned system

$$M^{-1}Gx \;=\; M^{-1}g, \tag{24}$$

with zero initial guess, all iterates belong to the Krylov subspace generated by the operator $M^{-1}G$ and the vector $M^{-1}g$, which is a subspace of the range of $M^{-1}G$. We denote the range of $M^{-1}G$ by $R_{M^{-1}G}$. The following two lemmas are also from [22].

Lemma 6. *Let* M^{-1} *be symmetric positive definite. Then the conjugate gradient method applied to solving (24) with zero initial guess cannot break down.*

Proof: We just need to show that for any $0 \neq x \in R_{M^{-1}G}$, $Gx \neq 0$. Let $0 \neq x = M^{-1}Gy$, for a certain $y \in X$ and $y \neq 0$. Then $Gx = GM^{-1}Gy$, which cannot be zero since $Gy \neq 0$ and $y^T GM^{-1}Gy \neq 0$. $\square$

Lemma 7. *Let* M^{-1} *be symmetric positive definite. For any* $x \in R_{M^{-1}G}$*,*

$$\langle Mx, x\rangle = \max_{y \in R_G, y \neq 0} \frac{\langle y, x\rangle^2}{\langle M^{-1}y, y\rangle}.$$

Proof: Denote the range of $M^{-\frac{1}{2}}G$ by $R_{M^{-1/2}G}$, then for any $\lambda \in R_{M^{-1}G}$

$$\begin{aligned} \langle Mx, x\rangle &= \left\langle M^{\frac{1}{2}}x, M^{\frac{1}{2}}x\right\rangle = \max_{z \in R_{M^{-1/2}G}, z \neq 0} \frac{\left\langle M^{\frac{1}{2}}x, z\right\rangle^2}{\langle z, z\rangle} \\ &= \max_{y \in R_G, y \neq 0} \frac{\left\langle M^{\frac{1}{2}}x, M^{-\frac{1}{2}}y\right\rangle^2}{\left\langle M^{-\frac{1}{2}}y, M^{-\frac{1}{2}}y\right\rangle} = \max_{y \in R_G, y \neq 0} \frac{\langle y, x\rangle^2}{\langle M^{-1}y, y\rangle}. \qquad \square \end{aligned}$$

6. The lumped preconditioner

The lumped preconditioner for solving (12) has been studied in [22] for the case of using continuous pressures in the finite element discretization. The preconditioner is given by

$$M_L^{-1} = \begin{bmatrix} \frac{1}{h^2} I_{p_\Gamma} & \\ & M_{L,\lambda}^{-1} \end{bmatrix},$$

where I_{p_Γ} is the identity matrix of the same length as p_Γ,

$$M_{L,\lambda}^{-1} = B_{\Delta,D} \widetilde{R}_\Delta \widetilde{A} \widetilde{R}_\Delta^T B_{\Delta,D}^T.$$

It has been proved in [22] that, for the case of using continuous pressures, the condition number of the preconditioned operator $M_L^{-1}G$ be bounded by $C\Phi_1(H,h)/\beta^2$, where $\Phi_1(H,h)$ is defined in Lemma 3 and β is the inf-sup constant specified in (4). The same bound also holds for more general cases where p_Γ is non-empty, including for using discontinuous pressures; see Section 4.1 and [22, Remark 6.10].

When p_Γ is empty, (12) becomes a system for the Lagrange multipliers only, which is the same as the one obtained in [15], cf. Section 4.2. The matrix G in (12) contains only the second diagonal block in (20), i.e.,

$$G = G_{\lambda\lambda} = B_\Delta \widetilde{R}_\Delta \widetilde{A}^{-1} \widetilde{R}_\Delta^T B_\Delta^T$$

and the lumped preconditioner M_L^{-1} simplifies to

$$M_L^{-1} = M_{L,\lambda}^{-1} = B_{\Delta,D} \widetilde{R}_\Delta \widetilde{A} \widetilde{R}_\Delta^T B_{\Delta,D}^T.$$

In this section, we establish the condition number bound for the case of empty p_Γ following the approach given in [22]. The bound we obtained here is the same as that obtained in [15], but this approach greatly simplifies the analysis given in [15]. We also note that this bound is also the same as that obtained for non-empty p_Γ in [22].

As discussed in [15] and [22], for using the lumped preconditioner, we only need use the first choice of the coarse level primal velocity space, i.e., $\mathbf{W}_\Pi$ contains only the subdomain corner velocities. We also note that when p_Γ is empty, p_I contains all the pressure degrees of freedom and the results in Section 5 are still valid. In fact the analysis given in the following is essentially the special case of the analysis in [22] when the blocks corresponding to p_Γ no longer exist. We have the following lemmas.

Lemma 8. *For any $w \in \widetilde{W}_0$, $\langle M_L^{-1} B_C w, B_C w\rangle \le \Phi_1(H,h) \left\langle \widetilde{A} w, w \right\rangle$, where $\Phi_1(H,h)$ is as defined in Lemma 3.*

Proof: For any given $w \in \widetilde{W}_0$, we have

$$\begin{aligned} \langle M_L^{-1} B_C w, B_C w\rangle &= \left(B_\Delta \widetilde{R}_\Delta w\right)^T M_L^{-1} B_\Delta \widetilde{R}_\Delta w \\ &= \left(B_\Delta \widetilde{R}_\Delta w\right)^T B_{\Delta,D} \widetilde{R}_\Delta \widetilde{A} \widetilde{R}_\Delta^T B_{\Delta,D}^T \left(B_\Delta \widetilde{R}_\Delta w\right) \\ &= \langle P_{D,1} w, P_{D,1} w\rangle_{\widetilde{A}} \le \Phi_1(H,h) \langle w, w\rangle_{\widetilde{A}}, \end{aligned} \tag{25}$$

where we have used Lemma 3 for the last inequality. □

Lemma 9. *For any given $y = g_\lambda \in R_G$, there exits $w \in \widetilde{W}_0$, such that $B_C w = y$, and $\left\langle \widetilde{A} w, w\right\rangle \le \frac{C}{\beta^2} \langle M_L^{-1} y, y\rangle$.*

Proof: Given $y = g_\lambda \in R_G$, let $\mathbf{u}_\Delta^{(I)} = B_{\Delta,D}^T g_\lambda$, $\mathbf{u}^{(I)} = \left(\mathbf{0},\ \mathbf{u}_\Delta^{(I)}, \mathbf{0}\right) \in \mathbf{W}_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi$, and $w^{(I)} = \left(\mathbf{0},\ 0,\ \mathbf{u}_\Delta^{(I)},\ \mathbf{0}\right) \in \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_\Pi$. We have

$$|\mathbf{u}^{(I)}|_{H^1}^2 = \left\langle A_{\Delta\Delta} \mathbf{u}_\Delta^{(I)}, \mathbf{u}_\Delta^{(I)} \right\rangle, \tag{26}$$

and

$$B_C w^{(I)} = \begin{bmatrix} 0 & 0 & B_\Delta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ 0 \\ B_{\Delta,D}^T g_\lambda \\ \mathbf{0} \end{bmatrix} = g_\lambda, \tag{27}$$

where we used the fact that $B_\Delta B_{\Delta,D}^T = I$.

We consider a solution to the following fully assembled system of linear equations of the form (3): find $\left(\mathbf{u}_I^{(II)},\ p_I^{(II)},\ \mathbf{u}_\Gamma^{(II)}\right) \in \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_\Gamma$, such that

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Gamma} \\ B_{II} & 0 & B_{I\Gamma} \\ A_{\Gamma I} & B_{I\Gamma}^T & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(II)} \\ p_I^{(II)} \\ \mathbf{u}_\Gamma^{(II)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -B_{I\Delta}\mathbf{u}_\Delta^{(I)} \\ \mathbf{0} \end{bmatrix}, \tag{28}$$

where a particular right-hand side is chosen. We first note that, since $g_\lambda \in R_G$, the right-hand side vector of the above system satisfies, cf. (14),

$$1_{p_I}^T \left(-B_{I\Delta}\mathbf{u}_\Delta^{(I)}\right) = -1_{p_I}^T B_{I\Delta} B_{\Delta,D}^T g_\lambda = 0,$$

i.e., it has a zero average, which implies the existence of solutions to the above system. Recall that here p_I contains all pressure degrees of freedom and there are no subdomain boundary pressure degrees of freedom.

Denote $\mathbf{u}^{(II)} = \left(\mathbf{u}_I^{(II)},\ \mathbf{u}_\Gamma^{(II)}\right) \in \mathbf{W}$. From the inf-sup stability of the original problem (3) and Lemma 5, we have

$$|\mathbf{u}^{(II)}|_{H^1}^2 \leq \frac{1}{\beta^2} \left\| B_{I\Delta}\mathbf{u}_\Delta^{(I)} \right\|_{Z^{-1}}^2. \tag{29}$$

The right-hand side of (29) can be bounded, using Lemma 1, as follows,

$$\begin{aligned} \left\| B_{I\Delta}\mathbf{u}_\Delta^{(I)} \right\|_{Z^{-1}}^2 &= \left\langle \widetilde{B}\mathbf{u}^{(I)}, \widetilde{B}\mathbf{u}^{(I)} \right\rangle_{Z^{-1}} = C \max_{q\in Q} \frac{\left\langle \widetilde{B}\mathbf{u}^{(I)}, q \right\rangle^2}{\langle q, q\rangle_Z} \\ &\leq C \max_{q\in Q} \frac{|\mathbf{u}^{(I)}|_{H^1}^2 \|q\|_{L^2}^2}{\|q\|_{L^2}^2} = C \left\langle A_{\Delta\Delta}\mathbf{u}_\Delta^{(I)}, \mathbf{u}_\Delta^{(I)} \right\rangle. \end{aligned} \tag{30}$$

Split the continuous subdomain boundary velocity $\mathbf{u}_\Gamma^{(II)}$ into the dual part $\mathbf{u}_\Delta^{(II)} \in \mathbf{W}_\Delta$ and the primal part $\mathbf{u}_\Pi^{(II)} \in \mathbf{W}_\Pi$, and denote $w^{(II)} = \left(\mathbf{u}_I^{(II)},\ p_I^{(II)},\ \mathbf{u}_\Delta^{(II)},\ \mathbf{u}_\Pi^{(II)}\right)$. We have, from (28),

$$\begin{bmatrix} B_{II} & 0 & B_{I\Delta} & B_{I\Pi} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(II)} \\ p_I^{(II)} \\ \mathbf{u}_\Delta^{(II)} \\ \mathbf{u}_\Pi^{(II)} \end{bmatrix} = -B_{I\Delta}\mathbf{u}_\Delta^{(I)}, \tag{31}$$

and

$$B_C w^{(II)} = \begin{bmatrix} 0 & 0 & B_\Delta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(II)} \\ p_I^{(II)} \\ \mathbf{u}_\Delta^{(II)} \\ \mathbf{u}_\Pi^{(II)} \end{bmatrix} = 0. \tag{32}$$

Let $w = w^{(I)} + w^{(II)}$. We can then see from (31) that $w \in \widetilde{W}_0$, cf. (18). We can also see from (27) and (32) that $B_C w = y$. Furthermore, by (19),

$$|w|^2_{\widetilde{A}} = |\mathbf{u}^{(I)} + \mathbf{u}^{(II)}|^2_{H^1} \leq |\mathbf{u}^{(I)}|^2_{H^1} + |\mathbf{u}^{(II)}|^2_{H^1} \leq \frac{C}{\beta^2} \left\langle A_{\Delta\Delta} \mathbf{u}_\Delta^{(I)}, \mathbf{u}_\Delta^{(I)} \right\rangle,$$

where we have used (26), (29) and (30) for the last inequality.

On the other hand, we have

$$\left\langle M_L^{-1} y, y \right\rangle = g_\lambda^T M_L^{-1} g_\lambda = g_\lambda^T B_{\Delta,D} \widetilde{R}_\Delta \widetilde{A} \widetilde{R}_\Delta^T B_{\Delta,D}^T g_\lambda = \left\langle A_{\Delta\Delta} \mathbf{u}_\Delta^{(I)}, \mathbf{u}_\Delta^{(I)} \right\rangle. \qquad \square$$

Theorem 1. *For all* $x = \lambda \in R_{M_L^{-1}G}$,

$$c\beta^2 \langle M_L x, x \rangle \leq \langle Gx, x \rangle \leq \Phi_1(H,h) \langle M_L x, x \rangle,$$

where $\Phi_1(H,h)$ *is as defined in Lemma 3,* β *is the inf-sup constant specified in (4).*

Proof: $\langle Gx, x \rangle = x^T B_C \widetilde{A}^{-1} B_C^T x = x^T B_C \widetilde{A}^{-1} \widetilde{A} \widetilde{A}^{-1} B_C^T x = \left\langle \widetilde{A}^{-1} B_C^T x, \widetilde{A}^{-1} B_C^T x \right\rangle_{\widetilde{A}}$.
Since $\widetilde{A}^{-1} B_C^T x \in \widetilde{W}_0$ and $\langle \cdot, \cdot \rangle_{\widetilde{A}}$ defines an inner product on $\widetilde{W}_0$, we have

$$\langle Gx, x \rangle = \max_{v \in \widetilde{W}_0, v \neq 0} \frac{\left\langle v, \widetilde{A}^{-1} B_C^T x \right\rangle_{\widetilde{A}}}{\langle v, v \rangle_{\widetilde{A}}} = \max_{v \in \widetilde{W}_0, v \neq 0} \frac{\langle B_C v, x \rangle^2}{\left\langle \widetilde{A} v, v \right\rangle}. \tag{33}$$

Lower bound: From Lemma 9, we know that for any given $y = g_\lambda \in R_G$, there exits $w \in \widetilde{W}_0$, such that $B_C w = y$ and $\left\langle \widetilde{A} w, w \right\rangle \leq \frac{C}{\beta^2} \left\langle M_L^{-1} y, y \right\rangle$. Then from (33), we have

$$\langle Gx, x \rangle \geq \frac{\langle B_C w, x \rangle^2}{\left\langle \widetilde{A} w, w \right\rangle} \geq c\beta^2 \frac{\langle y, x \rangle^2}{\left\langle M_L^{-1} y, y \right\rangle}.$$

Since y is arbitrary, using Lemma 7, we have

$$\langle Gx, x \rangle \geq c\beta^2 \max_{y \in R_G, y \neq 0} \frac{\langle y, x \rangle^2}{\left\langle M_L^{-1} y, y \right\rangle} = c\beta^2 \langle M_L x, x \rangle.$$

Upper bound: From (33), Lemmas 2, 8 and 7, we have

$$\begin{aligned} \langle Gx, x \rangle &\leq \Phi_1(H,h) \max_{v \in \widetilde{W}_0, v \neq 0} \frac{\langle B_C v, x \rangle^2}{\left\langle M_L^{-1} B_C v, B_C v \right\rangle} \\ &\leq \Phi_1(H,h) \max_{y \in R_G, y \neq 0} \frac{\langle y, x \rangle^2}{\left\langle M_L^{-1} y, y \right\rangle} = \Phi_1(H,h) \langle M_L x, x \rangle. \qquad \square \end{aligned}$$

7. The Dirichlet preconditioner

In the lumped preconditioner discussed in the previous section, the subdomain interface jump of the velocity component is extended by zero to the interior of subdomains. Comparing Lemmas 3 and 4, the discrete subdomain harmonic extension of the jump to the interior of subdomains has a better stability, which leads to the Dirichlet preconditioner discussed in this section.

Remark 3. *Subdomain discrete Stokes extensions, obtained by solving saddle-point problems, are used for the Dirichlet preconditioner studied in [21] for solving (12) with discontinuous pressures. In the Dirichlet preconditioner proposed in this section, the discrete subdomain harmonic extensions of the jump, obtained by solving symmetric positive definite problems, are used in each iteration. Even though these two extensions are spectrally equivalent, using the subdomain harmonic extensions is more efficient than solving indefinite subdomain problems.*

We define a Schur complement $S_{\Delta\Pi}^{(i)} : \mathbf{W}_{\Delta}^{(i)} \bigoplus \mathbf{W}_{\Pi}^{(i)} \to \mathbf{W}_{\Delta}^{(i)} \bigoplus \mathbf{W}_{\Pi}^{(i)}$. For any $\mathbf{u}_{\Gamma}^{(i)} = \left(\mathbf{u}_{\Delta}^{(i)},\ \mathbf{u}_{\Pi}^{(i)}\right) \in \mathbf{W}_{\Delta}^{(i)} \bigoplus \mathbf{W}_{\Pi}^{(i)}$, $S_{\Delta\Pi}^{(i)}\mathbf{u}_{\Gamma}^{(i)}$ is determined by

$$\begin{bmatrix} A_{II}^{(i)} & B_{II}^{(i)^T} & A_{I\Delta}^{(i)} & A_{I\Pi}^{(i)} & B_{\Gamma I}^{(i)^T} \\ B_{II}^{(i)} & 0 & B_{I\Delta}^{(i)} & B_{I\Pi}^{(i)} & 0 \\ A_{\Delta I}^{(i)} & B_{I\Delta}^{(i)^T} & A_{\Delta\Delta}^{(i)} & A_{\Delta\Pi}^{(i)} & B_{\Gamma\Delta}^{(i)^T} \\ A_{\Pi I}^{(i)} & B_{I\Pi}^{(i)^T} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} & B_{\Gamma\Pi}^{(i)^T} \\ B_{\Gamma I}^{(i)} & 0 & B_{\Gamma\Delta}^{(i)} & B_{\Gamma\Pi}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(i)} \\ p_I^{(i)} \\ \mathbf{u}_{\Delta}^{(i)} \\ \mathbf{u}_{\Pi}^{(i)} \\ p_{\Gamma}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ \left(S_{\Delta\Pi}^{(i)}\mathbf{u}_{\Gamma}^{(i)}\right)_{\Delta} \\ \left(S_{\Delta\Pi}^{(i)}\mathbf{u}_{\Gamma}^{(i)}\right)_{\Pi} \\ 0 \end{bmatrix}. \tag{34}$$

The solution of (34) will be needed below in the proof of Lemma 12. The subdomain incompressible Stokes problem (34) contains the subdomain constant pressure component. To guarantee the existence of its solution, the given subdomain boundary velocity $\mathbf{u}_{\Gamma}^{(i)} = \left(\mathbf{u}_{\Delta}^{(i)},\ \mathbf{u}_{\Pi}^{(i)}\right)$ need satisfy the divergence free condition $\int_{\partial\Omega_i} \mathbf{u}_{\Gamma}^{(i)} \cdot \mathbf{n} = 0$. In our applications, $\mathbf{u}_{\Gamma}^{(i)}$ will represent the jump of the subdomain interface velocity across the subdomain boundary and the coarse level component $\mathbf{u}_{\Pi}^{(i)}$ will always be zero. The divergence free condition will be satisfied if (6) is enforced. Due to this reason, for the condition number bound analysis for using the Dirichlet preconditioner, we assume that the second choice of the coarse level primal velocity space $\mathbf{W}_{\Pi}$ as described in Section 3 is used.

Remark 4. *The assumption that (6) is enforced is only required for the condition number bound analysis. When (6) is not enforced, e.g., when $\mathbf{W}_{\Pi}$ contains only the subdomain corner velocity variables, the Dirichlet preconditioner discussed below is still symmetric positive definite and the preconditioned conjugate gradient method can still be used, even though its condition number bound is no longer available.*

The following lemma is a well established result, cf. [2, Theorem 4.1] or [27, Lemma 3.1].

Lemma 10. *For all* $\mathbf{u}_{\Gamma}^{(i)} = \left(\mathbf{u}_{\Delta}^{(i)},\ \mathbf{u}_{\Pi}^{(i)}\right) \in \mathbf{W}_{\Delta}^{(i)} \bigoplus \mathbf{W}_{\Pi}^{(i)}$,

$$c\beta|\mathbf{u}_{\Gamma}^{(i)}|_{S_{\Delta\Pi}^{(i)}} \leq |\mathbf{u}_{\Gamma}^{(i)}|_{H^{1/2}(\partial\Omega^i)} \leq |\mathbf{u}_{\Gamma}^{(i)}|_{S_{\Delta\Pi}^{(i)}},$$

where β is the inf-sup constant specified in (4).

We also denote the direct sum of the discrete subdomain harmonic extension operators $H_\Delta^{(i)}, i = 1, \ldots, N$, defined in (21), by $H_\Delta : \mathbf{W}_\Delta \to \mathbf{W}_\Delta$.

In the following, the condition number bound for using the Dirichlet preconditioner is established for the case when p_Γ is non-empty. As discussed in Section 4.1, p_Γ in (12) is non-empty when either the continuous pressure is used in the finite element discretization, or the discontinuous pressure is used and p_Γ contains at least one pressure degree of freedom from each subdomain. In fact, the same condition number bound also holds for the case when p_Γ is empty, which will be discussed briefly at the end of this section.

We define $M_{D,\lambda}^{-1}$ by

$$M_{D,\lambda}^{-1} = B_{\Delta,D} H_\Delta B_{\Delta,D}^{\cdot} \tag{35}$$

and the Dirichlet preconditioner for solving (12) is

$$M_D^{-1} = \begin{bmatrix} \frac{1}{h^2} I_{p_\Gamma} & \\ & M_{D,\lambda}^{-1} \end{bmatrix}.$$

Lemma 11. *For any $w \in \widetilde{W}_0$, $\left\langle M_D^{-1} B_C w, B_C w \right\rangle \leq C \Phi_2(H,h) \left\langle \widetilde{A} w, w \right\rangle$, where $\Phi_2(H,h)$ is as defined in Lemma 4.*

Proof: Given $w = (\mathbf{w}_I,\ p_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{W}_0$, let $g_{p_\Gamma} = B_{\Gamma I} \mathbf{w}_I + B_{\Gamma\Delta} \mathbf{w}_\Delta + B_{\Gamma\Pi} \mathbf{w}_\Pi$. Similar to Lemma 8, we have from (22),

$$\begin{aligned} \left\langle M_D^{-1} B_C w, B_C w \right\rangle &= \frac{1}{h^2} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + (B_\Delta \mathbf{w}_\Delta)^T M_{D,\lambda}^{-1} B_\Delta \mathbf{w}_\Delta \\ &= \frac{1}{h^2} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + (B_\Delta \mathbf{w}_\Delta)^T B_{\Delta,D} H_\Delta B_{\Delta,D}^T B_\Delta \mathbf{w}_\Delta \\ &= \frac{1}{h^2} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + \langle P_{D,2} w, P_{D,2} w \rangle_{\widetilde{A}} \leq \frac{1}{h^2} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + \Phi_2(H,h) \langle w, w \rangle_{\widetilde{A}}, \end{aligned}$$

where we have used Lemma 4 for the last inequality. It is sufficient to bound the first term of the right-hand side in the above inequality.

We denote $\mathbf{w} = (\mathbf{w}_I,\ \mathbf{w}_\Delta,\ \mathbf{w}_\Pi) \in \widetilde{\mathbf{W}}$. Since $B_{II} \mathbf{w}_I + B_{I\Delta} \mathbf{w}_\Delta + B_{I\Pi} \mathbf{w}_\Pi = 0$, we have

$$\langle g_{p_\Gamma}, g_{p_\Gamma} \rangle = \begin{bmatrix} B_{II} \mathbf{w}_I + B_{I\Delta} \mathbf{w}_\Delta + B_{I\Pi} \mathbf{w}_\Pi \\ B_{\Gamma I} \mathbf{w}_I + B_{\Gamma\Delta} \mathbf{w}_\Delta + B_{\Gamma\Pi} \mathbf{w}_\Pi \end{bmatrix}^T \begin{bmatrix} B_{II} \mathbf{w}_I + B_{I\Delta} \mathbf{w}_\Delta + B_{I\Pi} \mathbf{w}_\Pi \\ B_{\Gamma I} \mathbf{w}_I + B_{\Gamma\Delta} \mathbf{w}_\Delta + B_{\Gamma\Pi} \mathbf{w}_\Pi \end{bmatrix} = \langle \widetilde{B} \mathbf{w}, \widetilde{B} \mathbf{w} \rangle,$$

where $\widetilde{B}$ is defined in (17). From (5) and the stability of $\widetilde{B}$, cf. Lemma 1, we have

$$\begin{aligned} \frac{1}{h^2} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle &= \frac{1}{h^2} \left\langle \widetilde{B} \mathbf{w}, \widetilde{B} \mathbf{w} \right\rangle \leq C \left\langle \widetilde{B} \mathbf{w}, \widetilde{B} \mathbf{w} \right\rangle_{Z^{-1}} = C \max_{q \in Q} \frac{\left\langle \widetilde{B} \mathbf{w}, q \right\rangle^2}{\langle q, q \rangle_Z} \\ &\leq C \max_{q \in Q} \frac{|\mathbf{w}|_{H^1}^2 \|q\|_{L^2}^2}{\|q\|_{L^2}^2} = C |\mathbf{w}|_{H^1}^2 = C \langle w, w \rangle_{\widetilde{A}}, \end{aligned} \tag{36}$$

where for the last equality, we used the fact that $B_{II} \mathbf{w}_I + B_{I\Delta} \mathbf{w}_\Delta + B_{I\Pi} \mathbf{w}_\Pi = 0$ and (19). □

Lemma 12. *Let the coarse level primal velocity space* $\mathbf{W}_\Pi$ *be chosen such that (6) is enforced. For any given* $y = (g_{p_\Gamma}, g_\lambda) \in R_G$*, there exits* $w \in \widetilde{W}_0$*, such that* $B_C w = y$*, and* $\left\langle \widetilde{A}w, w \right\rangle \leq \frac{C}{\beta^2} \left\langle M_D^{-1} y, y \right\rangle$.

Proof: Given $y = (g_{p_\Gamma}, g_\lambda) \in R_G$, let $\mathbf{u}_\Delta^{(I)} = B_{\Delta,D}^T g_\lambda$ and $\mathbf{u}_\Pi^{(I)} = \mathbf{0}$. On each subdomain Ω_i, denote $\left(\mathbf{u}_I^{(I,i)},\ p_I^{(I,i)},\ p_\Gamma^{(I,i)}\right)$ the part obtained through the solution of (34) with given subdomain boundary values $\mathbf{u}_\Delta^{(i)} = \mathbf{u}_\Delta^{(I,i)}$ and $\mathbf{u}_\Pi^{(i)} = \mathbf{0}$. Let $w^{(I,i)} = \left(\mathbf{u}_I^{(I,i)},\ p_I^{(I,i)},\ \mathbf{u}_\Delta^{(I,i)},\ \mathbf{u}_\Pi^{(I,i)}\right)$, the corresponding global vectors $w^{(I)} = \left(\mathbf{u}_I^{(I)},\ p_I^{(I)},\ \mathbf{u}_\Delta^{(I)},\ \mathbf{u}_\Pi^{(I)}\right)$, and $\mathbf{u}^{(I)} = \left(\mathbf{u}_I^{(I)},\ \mathbf{u}_\Delta^{(I)},\ \mathbf{u}_\Pi^{(I)}\right)$. Then we know from (34) that $w^{(I)} \in \widetilde{W}_0$, and

$$B_C w^{(I)} = \begin{bmatrix} B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} \\ 0 & 0 & B_\Delta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(I)} \\ p_I^{(I)} \\ \mathbf{u}_\Delta^{(I)} \\ \mathbf{u}_\Pi^{(I)} \end{bmatrix} = \begin{bmatrix} 0 \\ g_\lambda \end{bmatrix}, \tag{37}$$

where we used the fact that $B_\Delta B_{\Delta,D}^T = I$.

Using Lemma 10, we have

$$|\mathbf{u}^{(I,i)}|_{H^1(\Omega_i)}^2 = \left\| \begin{bmatrix} \mathbf{u}_\Delta^{(I,i)} \\ 0 \end{bmatrix} \right\|_{S_{\Delta\Pi}^{(i)}}^2 \leq \frac{C}{\beta^2} \left\| \begin{bmatrix} \mathbf{u}_\Delta^{(I,i)} \\ 0 \end{bmatrix} \right\|_{H^{1/2}(\partial\Omega^i)}^2,$$

and summing over the subdomains,

$$|\mathbf{u}^{(I)}|_{H^1}^2 \leq \frac{C}{\beta^2} \sum_{i=1}^N \left\| \begin{bmatrix} \mathbf{u}_\Delta^{(I,i)} \\ 0 \end{bmatrix} \right\|_{H^{1/2}(\partial\Omega^i)}^2. \tag{38}$$

We consider a solution to the following fully assembled system of linear equations of the form (3): find $\left(\mathbf{u}_I^{(II)},\ p_I^{(II)},\ \mathbf{u}_\Gamma^{(II)},\ p_\Gamma^{(II)}\right) \in \mathbf{W}_I \bigoplus Q_I \bigoplus \mathbf{W}_\Gamma \bigoplus Q_\Gamma$, such that

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Gamma} & B_{\Gamma I}^T \\ B_{II} & 0 & B_{I\Gamma} & 0 \\ A_{\Gamma I} & B_{I\Gamma}^T & A_{\Gamma\Gamma} & B_{\Gamma\Gamma}^T \\ B_{\Gamma I} & 0 & B_{\Gamma\Gamma} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(II)} \\ p_I^{(II)} \\ \mathbf{u}_\Gamma^{(II)} \\ p_\Gamma^{(II)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ \mathbf{0} \\ g_{p_\Gamma} \end{bmatrix}, \tag{39}$$

where a particular right-hand side is chosen. Since $y \in R_G$, we can see from (14) and (10), that $g_{p_\Gamma}^T 1_{p_\Gamma} = 0$. Therefore the above system has a solution.

Denote $\mathbf{u}^{(II)} = \left(\mathbf{u}_I^{(II)},\ \mathbf{u}_\Gamma^{(II)}\right)$. Then from Lemma 5 and (5), we have

$$|\mathbf{u}^{(II)}|_{H^1}^2 \leq \frac{1}{\beta^2} \left\| \begin{bmatrix} 0 \\ g_{p_\Gamma} \end{bmatrix} \right\|_{Z^{-1}}^2 \leq \frac{C}{\beta^2 h^2} \left\langle g_{p_\Gamma}, g_{p_\Gamma} \right\rangle. \tag{40}$$

Split the continuous subdomain boundary velocity $\mathbf{u}_\Gamma^{(II)}$ into the dual part $\mathbf{u}_\Delta^{(II)}$ and the primal part $\mathbf{u}_\Pi^{(II)}$, and denote $w^{(II)} = \left(\mathbf{u}_I^{(II)},\ p_I^{(II)},\ \mathbf{u}_\Delta^{(II)},\ \mathbf{u}_\Pi^{(II)}\right)$. Then we have from (39) that $w^{(II)} \in \widetilde{W}_0$ and

$$B_C w^{(II)} = \begin{bmatrix} B_{\Gamma I} & 0 & B_{\Gamma\Delta} & B_{\Gamma\Pi} \\ 0 & 0 & B_\Delta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(II)} \\ p_I^{(II)} \\ \mathbf{u}_\Delta^{(II)} \\ \mathbf{u}_\Pi^{(II)} \end{bmatrix} = \begin{bmatrix} g_{p_\Gamma} \\ 0 \end{bmatrix}. \tag{41}$$

Let $w = w^{(I)} + w^{(II)}$. We have $w \in \widetilde{W}_0$, $B_C w = y$, from (37) and (41), and from (19)

$$|w|^2_{\widetilde{A}} = |\mathbf{u}^{(I)} + \mathbf{u}^{(II)}|^2_{H^1} \le |\mathbf{u}^{(I)}|^2_{H^1} + |\mathbf{u}^{(II)}|^2_{H^1} \le \frac{C}{\beta^2} \sum_{i=1}^N \left|\left[\begin{matrix} \mathbf{u}_\Delta^{(I,i)} \\ 0 \end{matrix} \right]\right|^2_{H^{1/2}(\partial\Omega^i)} + \frac{C}{\beta^2 h^2} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle ,$$

where we have used (38) and (40) in the last inequality.

On the other hand, we have from (35)

$$\begin{aligned} \left\langle M_D^{-1} y, y \right\rangle &= \frac{1}{h^2} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + g_\lambda^T M_{D,\lambda}^{-1} g_\lambda \\ &= \frac{1}{h^2} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + g_\lambda^T \left(\sum_{i=1}^N B_{\Delta,D}^{(i)} H_\Delta^{(i)} {B_{\Delta,D}^{(i)}}^T \right) g_\lambda \\ &= \frac{1}{h^2} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + \sum_{i=1}^N \left|\left[\begin{matrix} \mathbf{u}_\Delta^{(I,i)} \\ 0 \end{matrix} \right]\right|^2_{H^{1/2}(\partial\Omega^i)} . \qquad \square \end{aligned}$$

With Lemmas 11 and 12, similar to the proof of Theorem 1, we have the following theorem.

Theorem 2. *Let the coarse level primal velocity space $\mathbf{W}_\Pi$ be chosen such that (6) is enforced. For all $x = (p_\Gamma,\ \lambda) \in R_{M_D^{-1}G}$,*

$$c\beta^2 \langle M_D x, x \rangle \le \langle Gx, x \rangle \le C\Phi_2(H,h) \langle M_D x, x \rangle ,$$

where $\Phi_2(H,h)$ is as defined in Lemma 4, β is the inf-sup constant specified in (4).

When p_Γ in (12) is empty, the matrix G contains only the second diagonal block in (20) and the Dirichlet preconditioner for (12) becomes only $M_{D,\lambda}^{-1}$, as given in (35), i.e.,

$$M_D^{-1} = M_{D,\lambda}^{-1} = B_{\Delta,D} H_\Delta B_{\Delta,D}^T .$$

Counterparts of Lemmas 11 and 12 for this case can be proved as well; in fact, their proofs are essentially the first half in the proofs of Lemmas 11 and 12, respectively. The condition number bound in Theorem 2 can then be established for this case in the same way.

8. Numerical experiments

We consider solving the incompressible Stokes problem (1) in the domain $\Omega = [0,1] \times [0,1]$. Zero Dirichlet boundary condition is used. The right-hand side function $\mathbf{f}$ is chosen such that the exact solution is

$$\mathbf{u} = \begin{bmatrix} \sin^3(\pi x)\sin^2(\pi y)\cos(\pi y) \\ -\sin^2(\pi x)\sin^3(\pi y)\cos(\pi x) \end{bmatrix} \quad \text{and} \quad p = x^2 - y^2.$$

Two mixed finite element discretizations, as shown on Figures 1 and 2 in Section 2, are used for the cases of discontinuous and continuous pressures, respectively. The preconditioned system (24) is solved by the preconditioned conjugate gradient method; the iteration is stopped when the L^2-norm of the residual is reduced by a factor of 10^{-6}.

In each of the following tables, we present the performance of three different variants of the FETI-DP algorithm represented under the same framework, as discussed in Sections 4.1 and 4.2: "continuous pressure" for the case when the continuous pressure is used in the algorithm and p_Γ contains all the subdomain boundary pressure degrees of freedom; "discontinuous pressure" for the case when the discontinuous pressure is used and p_Γ contains just one pressure degree of freedom from each subdomain; "p_Γ empty" for the case when the discontinuous pressure is used and p_Γ is chosen empty. For each case, the extreme eigenvalues and the iteration count for each experiment are shown. The two methods discussed in [21] and [15] solve the same system (12), and their only difference is in the implementation of multiplying G by a vector, cf. Remark 2. Therefore their convergence rates are the same, when they are equipped with the same type preconditioner, and their performance is reported under the case "p_Γ empty" in the tables.

Tables I and II show the performance of using the lumped preconditioner for different cases with two choices of the coarse level prime variables. In Table I, the first choice is used, namely only the subdomain corner velocities are taken as the coarse level primal variables. We can see that for each variant of the FETI-DP algorithm, the convergence rate is independent of the number of subdomains for fixed H/h; for fixed number of subdomains, the condition number grows presumably in the order of $(H/h)\,(1+\log(H/h))$ as established in Section 6. In Table II, we test the lumped preconditioner with the second choice of the coarse level primal space, namely it contains both the subdomain corner velocity variables and the edge-average velocity variables such that (6) is enforced, as discussed in Section 3. Even though the edge-average velocity variables are not required in the coarse level primal space for the lumped preconditioner case, including them improves the convergence rate for each method. We also observe from Tables I and II that performances of the three variants of the FETI-DP algorithm are quite similar, while the convergence when using discontinuous pressure and choosing p_Γ empty is a little faster than the other two cases.

Tables III and IV show the performance of using the Dirichlet preconditioner. In Table III, only the subdomain corner velocities are taken as the coarse level primal variables, for which the divergence free boundary condition (6) is not satisfied and no scalable condition number bound of the FETI-DP algorithm is available. Indeed Table III shows that for each variant of the FETI-DP algorithm, the convergence rate deteriorates with the increase of the number of subdomains when the subdomain problem size is fixed. In Table IV, both the subdomain corner and the edge-average velocity variables are taken as the coarse level primal variables such that (6) is enforced. We can see that for each variant of the FETI-DP algorithm, the convergence rate is independent of the number of subdomains for fixed H/h; for fixed number of subdomains, the condition number grows presumably in the order of $(1+\log(H/h))^2$ as established in Section 7.

Table I. Performance using lumped preconditioner M_L^{-1}, with corner primal variables.

		continuous pressure			discontinuous pressure			p_Γ empty		
H/h	#sub	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
8	4×4	0.35	8.92	21	0.48	7.93	22	0.56	7.37	20
	8×8	0.35	10.07	28	0.48	9.00	25	0.56	8.46	22
	16×16	0.35	10.23	29	0.48	9.20	25	0.56	8.71	22
	24×24	0.35	10.30	29	0.48	9.20	25	0.56	8.68	22
	32×32	0.35	10.33	29	0.48	9.21	25	0.56	8.68	22
#sub	H/h	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
8×8	4	0.30	4.22	21	0.41	3.91	19	0.54	3.75	16
	8	0.35	10.07	28	0.48	9.00	25	0.56	8.46	22
	16	0.35	24.22	36	0.49	21.39	36	0.56	20.30	33
	24	0.35	40.12	43	0.50	35.56	43	0.57	33.89	39
	32	0.35	57.15	50	0.50	50.87	50	0.57	48.62	45

Table II. Performance using lumped preconditioner M_L^{-1}, with corner and edge-average primal variables.

		continuous pressure			discontinuous pressure			p_Γ empty		
H/h	#sub	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
8	4×4	0.36	4.29	17	0.48	3.78	17	0.56	3.39	14
	8×8	0.36	5.29	21	0.49	4.47	18	0.56	4.01	16
	16×16	0.36	5.56	21	0.49	4.68	19	0.56	4.29	16
	24×24	0.36	5.61	21	0.50	4.77	19	0.55	4.42	16
	32×32	0.36	5.64	21	0.50	4.80	19	0.55	4.46	16
#sub	H/h	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
8×8	4	0.33	4.00	18	0.43	2.80	16	0.55	1.91	11
	8	0.36	5.29	21	0.49	4.47	18	0.56	4.01	16
	16	0.36	11.63	26	0.50	9.85	26	0.56	9.31	23
	24	0.36	18.67	31	0.50	16.05	32	0.57	15.36	29
	32	0.36	26.12	36	0.50	22.67	37	0.57	21.83	33

Table III. Performance using Dirichlet preconditioner M_D^{-1}, with corner primal variables.

		continuous pressure			discontinuous pressure			p_Γ empty		
H/h	#sub	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
8	4×4	0.32	3.11	18	0.38	2.83	16	0.42	1.87	12
	8×8	0.29	3.42	19	0.33	3.01	18	0.33	2.19	14
	16×16	0.25	3.52	21	0.28	3.09	19	0.28	2.30	16
	24×24	0.24	3.56	22	0.26	3.11	19	0.26	2.35	16
	32×32	0.23	3.57	22	0.25	3.12	20	0.25	2.37	17
#sub	H/h	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
8×8	4	0.27	3.82	22	0.33	2.73	18	0.37	1.70	13
	8	0.29	3.42	19	0.33	3.01	18	0.33	2.19	14
	16	0.30	4.00	21	0.32	3.39	19	0.32	2.84	16
	24	0.30	4.39	22	0.32	3.69	19	0.32	3.28	17
	32	0.31	4.71	23	0.32	3.95	20	0.32	3.62	19

Table IV. Performance using Dirichlet preconditioner M_D^{-1}, with corner and edge-average primal variables.

		continuous pressure			discontinuous pressure			p_Γ empty		
H/h	#sub	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
8	4×4	0.35	2.84	16	0.40	2.53	15	0.47	1.35	10
	8×8	0.35	2.96	16	0.41	2.65	15	0.47	1.59	10
	16×16	0.35	3.00	15	0.42	2.71	15	0.47	1.75	11
	24×24	0.35	3.02	15	0.42	2.74	15	0.47	1.79	11
	32×32	0.35	3.03	15	0.43	2.75	15	0.47	1.82	11
#sub	H/h	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter	λ_{min}	λ_{max}	iter
8×8	4	0.33	3.67	18	0.41	2.60	16	0.46	1.30	10
	8	0.35	2.96	16	0.41	2.65	15	0.47	1.59	10
	16	0.35	2.87	15	0.40	2.81	15	0.47	2.00	11
	24	0.34	2.99	16	0.40	2.97	16	0.50	2.30	12
	32	0.34	3.21	16	0.40	3.11	16	0.48	2.52	13

REFERENCES

1. Braess D. *Finite Elements*, 2nd ed.. Cambridge: Cambridge, UK, 2001.
2. Bramble JH, Pasciak JE. A domain decomposition technique for Stokes problems. *Applied Numerical Mathematics* 1989/90; **6**:251–261.
3. Dohrmann CR. Preconditioning of saddle point systems by substructuring and a penalty approach. In *Lecture Notes in Computational Science and Engineering* 2006; **55**:53–64.
4. Dohrmann CR. An approximate BDDC preconditioner. *Numerical Linear Algebra with Applications* 2007; **14**:149–168.
5. Dohrmann CR, Widlund OB. An overlapping Schwarz algorithm for almost incompressible elasticity. *SIAM Journal on Numerical Analysis* 2009; **47**:2897–2923.
6. Dohrmann CR, Widlund OB. Hybrid domain decomposition algorithms for compressible and almost incompressible elasticity. *International Journal for Numerical Methods in Engineering* 2010; **82**:157–183.
7. Farhat C, Lesoinne M, Le Tallec P, Pierson K, Rixen D. FETI-DP: A dual-primal unified FETI method – part I: A faster alternative to the two-level FETI method. *International Journal for Numerical Methods in Engineering* 2001; **50**:1523–1544.
8. Farhat C, Lesoinne M, Pierson K. A scalable dual-primal domain decomposition method. *Numerical Linear Algebra with Applications* 2000; **7**:687–714.
9. Farhat C. A Lagrange multiplier based divide and conquer finite element algorithm. *Computing Systems in Engineering* 1991; **2**:149–156.
10. Farhat C, Roux FX. A method of finite element tearing and interconnecting and its parallel solution algorithm. *International Journal for Numerical Methods in Engineering* 1991; **32**:1205–1227.
11. Farhat C, Roux FX. An unconventional domain decomposition method for an efficient parallel solution of large-scale finite element systems. *SIAM Journal on Scientific and Statistical Computing* 1992; **13**:379–396.
12. Farhat C, Mandel J, Roux FX. Optimal convergence properties of the FETI domain decomposition method. *Computer Methods in Applied Mechanics and Engineering* 1994; **115**:367–388.
13. Goldfeld P, Pavarino LF, Widlund OB. Balancing Neumann-Neumann preconditioners for mixed approximations of heterogeneous problems in linear elasticity. *Numerische Mathematik* 2003; **95**:283–324.
14. Kim HH, Lee CO. A Neumann-Dirichlet preconditioner for a FETI–DP formulation of the two-dimensional Stokes problem with mortar methods. *SIAM Journal on Scientific Computing* 2006; **28**:1133–1152.
15. Kim HH, Lee CO, Park EH. A FETI-DP formulation for the Stokes problem without primal pressure components. *SIAM Journal on Numerical Analysis* 2010; **47**:4142–4162.
16. Kim HH, Tu X. A three-level BDDC algorithm for mortar discretization. *SIAM Journal on Numerical Analysis* 2009; **47**:1576-1600.
17. Klawonn A, Pavarino LF. Overlapping Schwarz methods for mixed linear elasticity and Stokes problems. *Computer Methods in Applied Mechanics and Engineering* 1998; **165**:233–245.
18. Klawonn A, Rheinbach O. Inexact FETI-DP methods. *International Journal for Numerical Methods in Engineering* 2007; **69**:284–307.
19. Klawonn A, Widlund OB. Dual-Primal FETI methods for linear elasticity. *Communications on Pure and Applied Mathematics* 2006; **59**:1523–1572.
20. Klawonn A, Widlund OB, Dryja M. Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients. *SIAM Journal on Numerical Analysis* 2002; **40**:159–179.
21. Li J. A dual-primal FETI method for incompressible Stokes equations. *Numerische Mathematik* 2005; **102**:257–275.
22. Li J, Tu X. A Non-overlapping domain decomposition method for incompressible Stokes equation with continuous pressure. Submitted. Available at http://arxiv.org/abs/1204.1899.
23. Li J, Widlund OB. FETI–DP, BDDC, and block Cholesky methods. *International Journal for Numerical Methods in Engineering* 2006; **66**:250–271.
24. Li J, Widlund OB. BDDC algorithms for incompressible Stokes equations. *SIAM Journal on Numerical Analysis* 2006; **44**:2432–2455.
25. Li J, Widlund OB. On the use of inexact subdomain solvers for BDDC algorithms. *Computer Methods in Applied Mechanics and Engineering* 2007; **196**:1415–1428.
26. Mandel J, Tezaur R. On the convergence of a dual-primal substructuring method. *Numerische Mathematik* 2001; **88**:543–558.
27. Pavarino LF, Widlund OB. Balancing Neumann-Neumann methods for incompressible Stokes equations. *Communications on Pure and Applied Mathematics* 2002; **55**:302–335.
28. Pavarino LF, Widlund OB, Zampini S. BDDC preconditioners for spectral element discretizations of almost incompressible elasticity in three dimensions. *SIAM Journal on Scientific Computing* 2010; **32**:3604–3626.
29. Tu X. BDDC algorithm for a mixed formulation of flows in porous media. *Electronic Transactions on Numerical Analysis* 2005; **20**:164–179.

30. Tu X. A BDDC algorithm for flow in porous media with a hybrid finite element discretization. *Electronic Transactions on Numerical Analysis* 2007; **26**:146–160.
31. Tu X. Three-level BDDC. In *Lecture Notes in Computational Science and Engineering* 2007; **55**:437–444.
32. Tu X. Three-level BDDC in two dimensions. *International Journal for Numerical Methods in Engineering* 2007; **69**:33–59.
33. Tu X. Three-level BDDC in three dimensions. *SIAM Journal on Scientific Computing* 2007; **29**:1759–1780.
34. Tu X. A three-level BDDC algorithm for saddle point problems. *Numerische Mathematik* 2011; **119**:189–217.
35. Toselli A, Widlund OB. *Domain Decomposition Methods - Algorithms and Theory*. Springer-Verlag: New York, 2004.
36. Widlund OB. Iterative substructuring methods: Algorithms and theory for elliptic problems in the plane. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*. SIAM: Philadelphia, 1988.