Math 12002	Analytic Geometry and Calc I	Fall 2016
September 13, 2016	Exam 1	Matt Alexander

Name: Key Score: ______ /100

Please show **all** your work! Answers without supporting work will not be given credit. Write answers in spaces provided. You have 50 minutes to complete this exam.

1. (6 pts) Using complete sentences (and possibly some ϵ's and δ's), define the phrase "the limit of f as x approaches a is L."

For a function $f(x)$ defined on an interval around a. $\lim_{x\to a} f(x)=L$ if $\forall \varepsilon>0\ \exists \delta>0$ such that $|f(x)-L|<\varepsilon$ when $0<|x-a|<\delta$.

2. (6 pts ea.) Determine the following limits if they exist or are $\pm\infty$. **Justify your answers, and if a limit does not exist, explain why.**

(a) $\lim_{x\to 5} \frac{x^2-3x-10}{x^2-25} = \lim_{x\to 5} \frac{(x+2)(x-5)}{(x+5)(x-5)} = \lim_{x\to 5} \frac{x+2}{x+5} = \frac{7}{10}$

Answer: $\lim_{x\to 5} \frac{x^2-3x-10}{x^2-25} = \frac{7}{10}$

(b) $\lim_{x\to 0} \frac{\sqrt{x+25}-5}{x}$

Notice $\frac{\sqrt{x+25}-5}{x} \cdot \frac{\sqrt{x+25}+5}{\sqrt{x+25}+5} = \frac{x+25-25}{x\cdot(\sqrt{x+25}+5)} = \frac{x}{x(\sqrt{x+25}+5)}$

Since $x\neq 0$ $\lim_{x\to 0} \frac{\sqrt{x+25}-5}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+25}+5} = \frac{1}{\sqrt{25}+5} = \frac{1}{10}$

Answer: So $\lim_{x\to 0} \frac{\sqrt{x+25}-5}{x} = \frac{1}{10}$

(c) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} 3x+2 & \text{if } x<1 \\ x^5+2x^2 & \text{if } x\geq 1 \end{cases}$

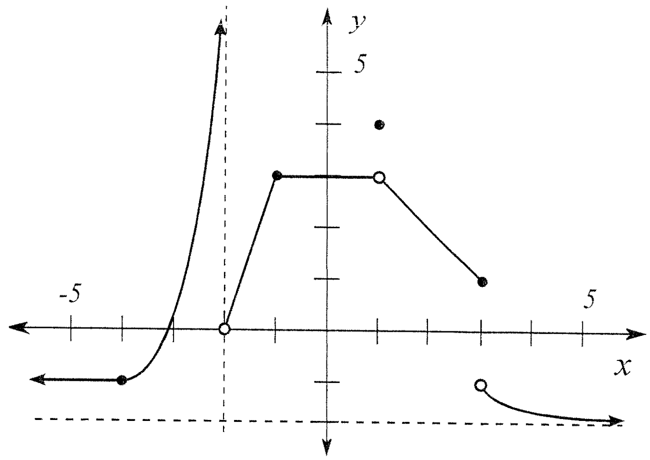
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^5+2x^2) = 1+2(1) = 3$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x+2) = 3(1)+2 = 5$

Since $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$

Answer: $\lim_{x\to 1} f(x)$ DNE

3. (22 pts) The function f is depicted below.



(a) Determine the following limits or state "does not exist". ("dne").

$\lim_{x\to-4^-} f(x) =$ -1	$\lim_{x\to-2^-} f(x) =$ ∞	$\lim_{x\to1^-} f(x) =$ 3	$\lim_{x\to3^-} f(x) =$ 1
$\lim_{x\to-4^+} f(x) =$ -1	$\lim_{x\to-2^+} f(x) =$ 0	$\lim_{x\to1^+} f(x) =$ 3	$\lim_{x\to3^+} f(x) =$ -1
$\lim_{x\to-4} f(x) =$ -1	$\lim_{x\to-2} f(x) =$ DNE	$\lim_{x\to1} f(x) =$ 3	$\lim_{x\to3} f(x) =$ DNE
$f(-4) =$ -1	$f(-2) =$ DNE	$f(1) =$ 4	$f(3) =$ 1

(b) For each of the following answer yes or no. If no, explain.

i. Is f continuous at $x = 1$?

No, $\lim_{x\to1} f(x) = 3$ but $f(1) = 4$.

ii. Is f continuous at $x = 3$?

No, $\lim_{x\to3} f(x)$ DNE

(c) Find each of the following limits.

$\lim_{x\to\infty} f(x) =$ -2 $\qquad$ $\lim_{x\to-\infty} f(x) =$ -1

4. (6 pts) Suppose that $\lim_{x\to2} f(x) = -1$ and $\lim_{x\to2} g(x) = 2$. Find $\lim_{x\to2} \frac{2f(x) - g(x)^2}{f(x) + f(x)g(x)}$. Remember to show work.

$$\lim_{x\to2} \frac{2f(x)-g(x)^2}{f(x)+f(x)g(x)} = \frac{2\lim_{x\to2} f(x) - \left(\lim_{x\to2} g(x)\right)^2}{\lim_{x\to2} f(x) + \lim_{x\to2} f(x) \cdot \lim_{x\to2} g(x)} = \frac{2(-1)-(2)^2}{-1+(-1)(2)} = \frac{-2-4}{-1-2} = \frac{-6}{-3} = 2$$

Answer: $\lim_{x\to2} \frac{2f(x)-g(x)^2}{f(x)+f(x)g(x)} = 2$

5. (6 pts) Suppose we have a function $f(x)$ such that $-|x| \le f(x) \le x^2$ for all x between -1 and 1. Find $\lim_{x\to 0} f(x)$ and justify using the appropriate theorem.

Since $-|x| \le f(x) \le x^2$ and $\lim_{x\to 0} -|x| = \lim_{x\to 0} x^2 = 0$ we have by the squeeze theorem that $\lim_{x\to 0} f(x) = 0$.

Answer: $\lim_{x\to 0} f(x) = 0$

6. (6 pts ea.) Determine the following limits if they exist or are $\pm\infty$. **Justify your answers, and if a limit does not exist, explain why.**

(a) $\lim_{x\to 4^+} \dfrac{x+3}{(x-4)(x-2)}$

Since $(x-4) \to 0$ as $x \to 4$ we have that the denominator goes to 0 but the numerator does not. Hence we have a vertical asymptote. So we use a sign chart to determine $\pm\infty$.

$x+3$: $-$ -3 $+$ $+$ $+$

$x-4$: $-$ $-$ $-$ 4 $+$

$x-2$: $-$ $-$ 2 $+$ $+$

$-$ -3 $+$ 2 $-$ 4 $+$

So by the sign chart

Answer: $\lim_{x\to 4^+} \dfrac{x+3}{(x-4)(x-2)} = \infty$

(b) $\lim_{x\to+\infty} \dfrac{3x^8 - 2x^2 + 5}{7 - 5x^5 + 4x^8} = \lim_{x\to\infty} \dfrac{x^8\left(3 - \frac{2}{x^6} + \frac{5}{x^8}\right)}{x^8\left(\frac{7}{x^8} - \frac{5}{x^3} + 4\right)} = \lim_{x\to\infty} \dfrac{3 - \frac{2}{x^6} + \frac{5}{x^8}}{\frac{7}{x^8} - \frac{5}{x^3} + 4} = \dfrac{3-0+0}{0-0+4} = \dfrac{3}{4}$

Answer: $\lim_{x\to\infty} \dfrac{3x^8 - 2x^2 + 5}{7 - 5x^5 + 4x^8} = \dfrac{3}{4}$

7. (10 pts) Find all horizontal and vertical asymptotes of $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 2x}$. **Justify your answers.**

$f(x) = \dfrac{(x-2)(x-3)}{x(x-2)} = \dfrac{x-3}{x}$, $x \ne 0$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{x^2 - 5x + 6}{x^2 - 2x} = \lim_{x\to\infty} \dfrac{\cancel{x^2}\left(1 - \frac{5}{x} + \frac{6}{x^2}\right)}{\cancel{x^2}\left(1 - \frac{2}{x}\right)} \overset{\text{since } x \ne 0}{=} \dfrac{1-0+0}{1-0} = 1$

Since $f(x) = \dfrac{x-3}{x}$, $x \ne 0$ we know that $\lim_{x\to 2} f(x) = -\frac{1}{2}$ and so there is no V.A. at $x=2$.

Since $x \to 0$ as $x \to 0$ the denominator $\to 0$ but the numerator does not. Hence there is a V.A. at $x=0$

So we have a H.A. at $y=1$ since rational functions can have only one H.A.

Answer: H.A. at $y=1$ and V.A. at $x=0$

Cont.

8. (2 pts) Spell the word "asymptote."

Answer: Asymptote

9. (6 pts ea.) State whether the following functions are continuous at the given points. Justify your answers using either a theorem or the definition of continuity at a point.

(a) $f(x) = \begin{cases} \dfrac{x^2-2x-8}{x^2-7x+12} & \text{if } x \neq 4 \\ 6 & \text{if } x = 4 \end{cases}$ at the point $x = 4$.

$$\frac{x^2-2x-8}{x^2-7x+12} = \frac{(x+2)(x-4)}{(x-3)(x-4)} = \frac{x+2}{x-3}$$

So since $x \neq 4$, $\lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{x+2}{x-3} = \frac{6}{1} = 6$,

and $f(4) = 6$, then

Answer: ~~the~~ $f(x)$ is continuous at $x=4$

(b) $f(x) = \begin{cases} 3x^2-4x+2 & \text{if } x < -1 \\ 5x^3-6x^2+3 & \text{if } x \geq -1 \end{cases}$ at the point $x = -1$.

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (3x^2-4x+2) = 3+4+2 = 9$ (continuous from left)

and $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (5x^3-6x^2+3) = -5-6+3 = -8$ (continuous from right)

Then since $\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$, $\lim_{x\to -1} f(x)$ DNE and

Answer: $f(x)$ is not continuous at $x=-1$

10. (6 pts) Use the intermediate value theorem to explain why there is a solution to the equation $\sin(\theta) = \cos(\theta)$ between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Notice the function $f(\theta) = \cos(\theta) - \sin(\theta)$ is continuous for all θ and $f(0) = 1$ and $f(\frac{\pi}{2}) = -1$. Hence by the IVT there exist a c between 0 and $\frac{\pi}{2}$ such that $f(c) = 0$ or $\sin(c) = \cos(c)$.