Math 12002	Analytic Geometry and Calc I	Fall 2016
October 4, 2016	Exam 2	Matt Alexander

Name: Key   Score: ______ /100

> Please show **all** your work! Answers without supporting work will not be given credit. Write answers in spaces provided. You have 50 minutes to complete this exam.

1. (a) (2 points) State the limit definition of the derivative of a function at a point. (i.e. $f'(a)$)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(b) (8 points) If $f(x) = x^2 + 3x - 5$, find $f'(-6)$ using only the limit definition of the derivative.

$$f'(-6) = \lim_{h \to 0} \frac{f(-6+h) - f(-6)}{h} = \lim_{h \to 0} \frac{(-6+h)^2 + 3(-6+h) - 5 - [(-6)^2 + 3(-6) - 5]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{(-6)^2} - 12h + h^2 + \cancel{(3)(-6)} + 3h \cancel{- 5} \cancel{- (-6)^2} \cancel{- 3(-6)} \cancel{+ 5}}{h}$$

$$= \lim_{h \to 0} \frac{-9h + h^2}{h} = \lim_{h \to 0} \frac{h(-9+h)}{h} = \lim_{h \to 0} -9 + h = -9$$

(c) (5 points) Find the equation of the tangent line to the graph of $f(x) = x^2 + 3x - 5$ at the point $(-6, 13)$. $m = -9$ from part (b) then

$y - 13 = -9(x + 6)$

$y = -9x - 54 + 13$

Answer: $y = -9x - 41$ ______

2. (5 points) Let $y = f(x)$ be a function of x. Explain why the instantaneous rate of change of y with respect to x at the point $x = a$ is given by the derivative of f at a. (In particular, discuss average rate of change and its relationship to instantaneous rate of change).

The average rate of change between two points on a graph $x = a$ and $x = a + h$ is given by $\frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$. Then as $a+h$ approaches a by $h \to 0$ the average rate of change approaches the instantaneous rate of change. That is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ is the instantaneous rate of change and also the definition of the derivative.

3. (24 points) Find the derivatives of the following functions. *Do not simplify your answers.* Clearly mark your answer.

(a) $f(x) = 8x^4 + \sqrt[4]{x^3} + \dfrac{2}{x^3} + 2^2$

$f'(x) = 32x^3 + \frac{3}{4}x^{-\frac{1}{4}} - 6x^{-4}$

(b) $g(x) = (3x^2 + x)\tan(x)$

$g'(x) = (6x+1)\tan(x) + (3x^2+x)\sec^2(x)$

(c) $h(x) = \dfrac{5 + \cos x}{x^4 + \csc x}$

$h'(x) = \dfrac{-\sin(x)\,(x^4+\csc(x)) - (5+\cos(x))(4x^3 - \csc(x)\cot(x))}{(x^4+\csc(x))^2}$

(d) $F(x) = (8x^3 + \cos x)^{10}$

$F'(x) = 10(8x^3+\cos(x))^9(24x^2 - \sin(x))$

4. (7 points) Find the second derivative, f'', of $f(x) = (5x + 4)^4$.

$f'(x) = 4(5x+4)^3 \cdot 5 = 20(5x+4)^3$

$f''(x) = 60(5x+4)^2 \cdot 5$

Answer: $f''(x) = 300(5x+4)^2$

5. (5 points) Let f and g be two differentiable functions such that $f(2) = 3$, $f'(2) = 5$, $g(2) = 6$, and $g'(2) = -1$. If $h(x) = 2f(x)g(x) - g(x)^2$ find $h'(2)$.

$h'(2) = 2(f'(2)g(2) + g'(2)f(2)) - 2g(2)g'(2)$

$= 2(5 \cdot 6 + (-1)3) - 2(6)(-1)$

$= 2(30-3) + 12$

$= 2(27) + 12 = 54 + 12 = 66$

Answer: $h'(2) = 66$

Cont.

6. (7 points) Find the slope of the line tangent to the ellipse $9x^2 + 16y^2 = 25$ at the point $(1, 1)$.

$18x + 32yy' = 0$

$y' = -\frac{18x}{32y} = -\frac{9x}{16y}$

$y'\big|_{(1,1)} = -\frac{9}{16}$

Answer: The slope of the tangent line is $-\frac{9}{16}$

7. (7 points) Find y' (the derivative of y with respect to x) if $3x + y^2 + \cos y + x^2y^3 = 2$.

$3 + 2yy' + (-\sin y)\cdot y' + 2xy^3 + 3x^2y^2y' = 0$

$y'(2y - \sin(y) + 3x^2y^2) = -2xy^3 - 3$

$y' = \frac{-2xy^3 - 3}{2y - \sin(y) + 3x^2y^2}$

Answer: $y' = \frac{-2xy^3 - 3}{2y - \sin(y) + 3x^2y^2}$

8. (10 points) A particle moves along a straight line and its position (in feet) at time t (in seconds) is given by $s(t) = 5t^2 - 40t + 60$.

(a) Find the velocity and acceleration of the particle at time $t = 3$

$s'(t) = 10t - 40$

$s''(t) = 10$

v(t)= $10t - 40$, $v(3) = -10$ ft per second

a(t)= 10, $a(3) = 10$ ft per second^2

(b) Determine the time t at which the particle is **not** moving.

$s'(t) = 10t - 40 = 0$

$t = 4$

Answer: $t = 4$ seconds

Cont.

9. (10 points) A large spherical balloon is being inflated with helium at a rate of 200 cubic meters of air per minute. Let r be the radius of the balloon, V be its volume, and t be time. (**Note:** $V = \frac{4}{3}\pi r^3$)

(a) What is the rate of change of the radius of the balloon when the radius is 2 meters?

$V = \frac{4}{3}\pi r^3$ So $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$ by chain rule

$\frac{dV}{dr} = 4\pi r^2$ $200 \frac{m^3}{min} = 4\pi r^2 \cdot \frac{dr}{dt}$

$\frac{dV}{dt} = 200\ m^3/min$ or $\left.\frac{dr}{dt}\right|_{r=2} = \frac{200\ m^3}{4\pi (2m)^2\ min} = \frac{12.5}{\pi}\ m/min$ or $\frac{25}{2\pi}\ m/min$

Answer: $\frac{12.5}{\pi}$ meters per minute

(b) What is the rate of change of the radius of the balloon when the radius is 10 meters?

From above $\left.\frac{dr}{dt}\right|_{r=10} = \frac{200\ m^3}{4\pi (10m)^2\ min} = \frac{1}{2\pi}\ m/min$

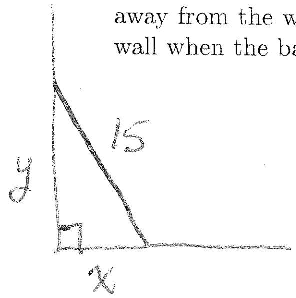
Answer: $\frac{1}{2\pi}$ meters per minute

(c) Using complete sentences, compare these two values give an explanation for why they are different.

When $r=2$ the balloon is small and the volume small compared to the rate that it is filling, but when $r=10$ the balloon is larger and its volume is much larger than the rate it is filling. The change in radius slows as the balloon grows since the volume grows much more quickly than the radius (cubic versus linear).

Cont.

10. (10 points) A ladder 15 feet long rests against a vertical wall. If the bottom of the ladder is pulled away from the wall at a speed of 3 feet per second, how fast is the top of the ladder sliding down the wall when the base is 9 feet from the wall?



Let x be the distance of the bottom of the ladder to the wall
Let y be the distance of the top of the ladder to the ground

By Pythagorean theorem $x^2+y^2=(15)^2$

By implicit differentiation $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$ or $\frac{dy}{dt}=\frac{-x\frac{dx}{dt}}{y}$

When $x=9$ $y=\sqrt{15^2-9^2}=12$ by Pythagorean again.

So $\left.\frac{dy}{dt}\right|_{x=9\ y=12} = \frac{-9\cdot 3 \text{ ft/sec}}{12} = -\frac{9}{4}$ ft/sec

Answer: The top of the ladder is sliding down at a rate of $\frac{9}{4}$ ft/sec

The End.