Math 12002	Analytic Geometry and Calc I	Fall 2016
October 20, 2016	Exam 3	Matt Alexander

Name: Key Score: ____/100

> Please show **all** your work! Answers without supporting work will not be given credit. Write answers in spaces provided. You have 60 minutes to complete this exam.

1. (10 points) If $f'(x) \le 10$ for all x and $f(2) = 7$, what is the largest possible value of $f(5)$?

By MVT $\exists\, c \in (2,5)$ such that

$$f(5) - f(2) = f'(c)(5-2)$$

So $f(5) = f'(c)(3) + 7$

Then $f(5) \le 10 \cdot 3 + 7 = 37$ since $f'(c) \le 10$.

Therefor the largest possible value of $f(5)$ is 37

2. (8 points) Does there exist a function f such that $f'(x) \le 3$ for all x, $f(-2) = 150$, and $f(3) = 200$? Justify your answer.

If such a function exist then by MVT $\exists\, c \in (-2,3)$ such that

$$f'(c) = \frac{f(3) - f(-2)}{3-(-2)} = \frac{200-150}{5} = 10$$

But $f'(x) \le 3$ for all x which contradicts what we found so no such function can exist.

3. (8 points) Explain, using either Rolle's theorem or MVT, why a ball launched into the air that falls back to the ground must have a time during its flight when its vertical speed is 0.

Let $h(t)$ be the height of the ball at time t seconds after it is launched. Then h is a continuous and differentiable function and $f(0) = f(t_1) = 0$ where t_1 is the time the ball hits the ground. Then by Rolles theorem there exists a time $t_0 \in (0, t_1)$ such that $h'(t_0) = 0$ where $h'(t)$ is the vertical speed of the ball at time t.

4. (10 points) Let $f(x) = x^5 + 5x^4 + 5x^3$. Find all critical points, intervals where f is increasing, intervals where f is decreasing, and the x values of all local maxima and minima.

2 $f'(x) = 5x^4 + 20x^3 + 15x^2 = 5x^2(x^2+4x+3) = 0$

$5x^2(x+3)(x+1) = 0$

2 Crit #'s $x=0, x=-3, x=-1$

2 Sign chart f': $+$ on $(-\infty,-3)$, $-$ on $(-3,-1)$, $+$ on $(-1,0)$, $+$ on $(0,\infty)$; marks at -3, -1, 0

f is increasing on $(-\infty,-3) \cup (-1,0) \cup (0,\infty)$

4 f is decreasing on $(-3,-1)$

f has a local max when $x=-3$

f has a local min when $x=-1$

5. (10 points) Let $g(x) = x^3 - 2x^4$. Find all intervals where g is concave up, intervals where g is concave down, and the x values of all points of inflection.

2 $g'(x) = 3x^2 - 8x^3$ $\quad g''(x) = 3\cdot 2x - 8\cdot 3x^2 = 6x(1-4x) = 0$

2 possible inflection points when $x=0$ or $x=\frac{1}{4}$

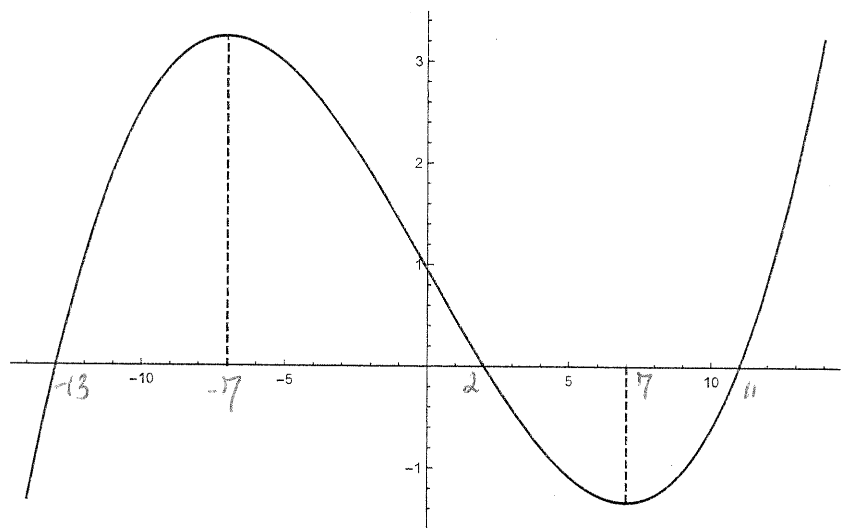
2 Sign chart g'': $-$ on $(-\infty,0)$, $+$ on $(0,\frac{1}{4})$, $-$ on $(\frac{1}{4},\infty)$; marks at 0, $1/4$

2 g is concave up on $(0,\frac{1}{4})$

g is concave down on $(-\infty,0) \cup (\frac{1}{4},\infty)$

2 g has inflection points when $x=0$ and $x=\frac{1}{4}$

6. (14 points) The graph below is the graph of the DERIVATIVE, $f'(x)$, of a function $y = f(x)$



[NOTE: The graph above is of the derivative f' of f. The questions below refer to f, not to f'.]

(a) Determine the intervals where f is increasing and where f is decreasing.

f is increasing on $(-13, 2) \cup (11, \infty)$

4

f is decreasing on $(-\infty, -13) \cup (2, 11)$

(b) Determine the intervals where f is concave up and where f is concave down.

f is concave up on $(-\infty, -7) \cup (7, \infty)$

4

f is concave down on $(-7, 7)$

(c) Find the x values of all local maxima and minima of f (State whether each is a local maximum or local minimum.)

Local max at $x = 2$

4

Local min at $x = -13$; $x = 11$

(d) Find the x values of all inflection points of f.

2

Inflection points when $x = -7$ and $x = 7$

7. (14 points) Let $f(x) = \frac{4(x^2+3x+1)}{(x+1)^2}$, so that $f'(x) = \frac{4(1-x)}{(x+1)^3}$, and $f''(x) = \frac{8(x-2)}{(x+1)^4}$.
Then $\lim_{x\to+\infty} f(x) = 4$, $\lim_{x\to-\infty} f(x) = 4$, and $\lim_{x\to-1} f(x) = -\infty$, and the following points are on the graph: $\{(1,5), (2,4.8), (0,4), (-2.6,0), (-0.38,0)\}$ (Take the values as I've rounded them)

(a) Given that f is increasing on $(-1,1)$ and decreasing on $(-\infty,-1)\cup(1,\infty)$, draw the appropriate sign chart for f'.

2

f' — -1 + 1 —

(b) Find all local maxima and minima of f.

2 Local max at $(1,5)$

(c) Given that f is concave-up on $(2,\infty)$ and concave-down on $(-\infty,-1)\cup(-1,2)$, draw the appropriate sign chart for f''.

2

f'' — -1 — 2 +

(d) Find all inflection points of f.

1 Inflection point at $(2, 4.8)$

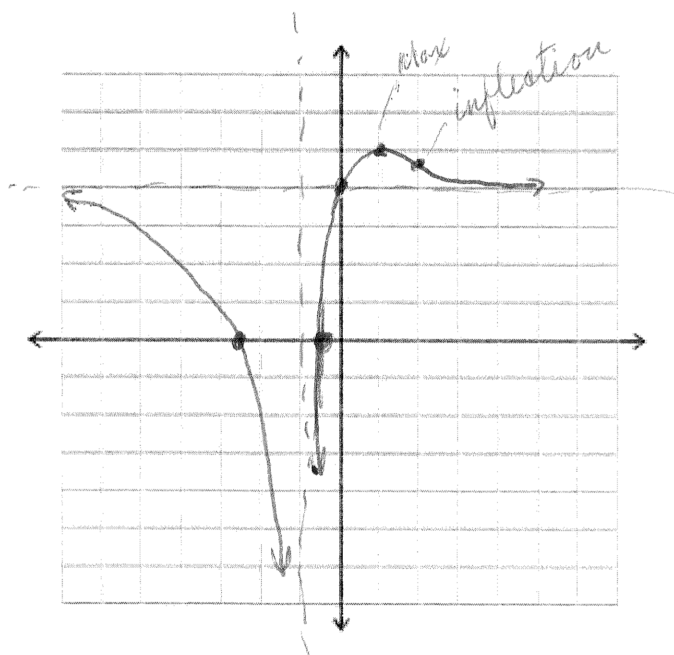
(e) Determine the vertical and horizontal asymptotes of f, if any. Justify your answer by the given information. V.A. of $x=-1$ since $\lim_{x\to-1} f(x) = -\infty$

3

H.A. of $y=4$ since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 4$

(f) Sketch the graph of f on the set of axes below, clearly indicating all of the information given and obtained above.

4



8. (10 points) Find the absolute maximum and absolute minimum values of the function $f(x) = x^3 - 6x^2 = x^2(x-6)$ on the closed interval $[-1, 7]$

2 $f'(x) = 3x^2 - 12x = 3x(x-4)$ 2 Crit #s $x=0$ $x=4$

4
$f(-1) = (-1)^2(-1-6) = -7$
$f(0) = 0^2(0-6) = 0$
$f(4) = 4^2(4-6) = 16(-2) = -32$
$f(7) = 7^2(7-6) = 49$

So by EVT the absolute:
2 Max is 49 when $x=7$
Min is -32 when $x=4$

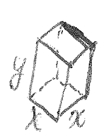
9. (10 points) If the product of two positive numbers is 9, what is smallest possible value of the sum of their squares?

1 Let x be the ~~first~~ number, y the second. $x, y > 0$
3 Then $xy = 9$ and $S = x^2 + y^2$ so $y = \frac{9}{x}$ and

3
$S(x) = x^2 + \left(\frac{9}{x}\right)^2$
$S'(x) = 2x + 9^2 \cdot 2(-1)\frac{1}{x^3} = 0$
$\Rightarrow 2x - 2 \cdot 9^2\left(\frac{1}{x^3}\right) = 0$
$2x = 2 \cdot 9^2 \cdot \frac{1}{x^3}$
$x^4 = 9^2 = 3^4$ so $x = \pm 3$

3
-3 is discarded since $x > 0$
$S(3) = 18$ while $S(0) = S(9) = 9^2 = 81$
So the smallest possible sum is 18 by EVT.

10. (6 points) If a box with a square bottom and open top is to be made from 48 sq. ft. of material, find the function $V(x)$ that gives the volume of the box in terms of one side length. [Note: This is asking you to set up the beginning of an optimization problem. Do not solve]



2 Let x be the length of the bottom side of the box and y be the height as labelled. Then the total material needed for the box is the surface area of the sides and bottom. So

2 $4xy + x^2 = 48$ and $V = x^2 y$. Then $y = \frac{48 - x^2}{4x}$ gives

2 $V(x) = x^2\left(\frac{48 - x^2}{4x}\right)$

The End.