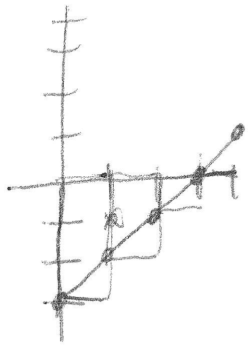
Math 12002	Analytic Geometry and Calc I	Fall 2016
November 15, 2016	Exam 4	Matt Alexander

Name: Key Score: ______ /100

> Please show **all** your work! Answers without supporting work will not be given credit.
> Write answers in spaces provided. You have 50 minutes to complete this exam.

1. (6 points) Compute a Riemann sum using left end points and 4 approximating rectangles for the function $f(x) = x - 3$ on the interval $[0, 4]$



$$S_4 = 1\cdot(-3) + 1\cdot(-2) + 1\cdot(-1) + 0\cdot 1$$
$$= -3 - 2 - 1 = -6$$

Answer: $S_4 = -6$

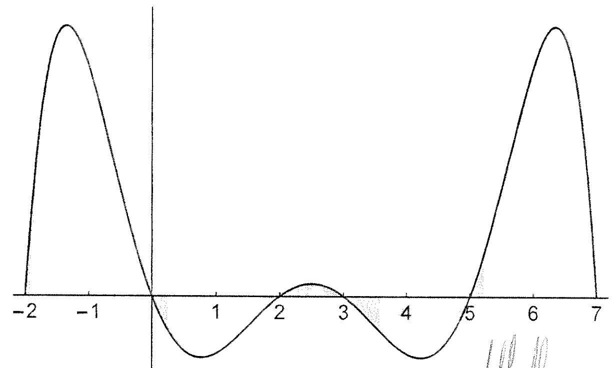
2. (4 points) Write the limit definition of the definite integral of the function $f(x) = x^2 - 2$ on the interval $[2, 5]$, using the standard test values and regular intervals. **Do not compute**.

$\Delta x = \frac{5-2}{n} = \frac{3}{n}$

$x_i = a + \Delta x \cdot i = 2 + \frac{3i}{n}$

Answer: $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(2 + \frac{3i}{n}\right)^2 - 2\right)\frac{3}{n}$

3. (6 points) Express the area of the shaded region in terms of definite integrals of the function f whose graph is given below.



$\int_{-2}^{7} |f(x)|dx$

All the same

or

Answer: $\int_{-2}^{0} f(x)dx - \int_{0}^{2} f(x)dx + \int_{2}^{3} f(x)dx - \int_{3}^{5} f(x)dx + \int_{5}^{7} f(x)dx$

or $2\int_{-2}^{0} f(x)dx - 2\int_{0}^{2} f(x)dx + \int_{2}^{3} f(x)dx$

4. (10 points) Assume $\int_2^6 f(x)\,dx = 2$ and $\int_2^6 g(x)\,dx = 10$. Compute the following integrals, if possible. **If not enough information is given to compute an integral, indicate this.**

(a) $\displaystyle\int_2^6 \frac{f(x)}{g(x)}\,dx =$

Answer: Not possible

(b) $\displaystyle\int_2^6 [f(x) - g(x)]\,dx =$

$\int_2^6 f(x)\,dx - \int_2^6 g(x)\,dx = 2 - 10$

Answer: -8

(c) $\displaystyle\int_2^6 [3f(x) + 1]\,dx =$

$3\int_2^6 f(x)\,dx + \int_2^6 1\,dx = 3(2) + (6-2)$

$= 6 + 4$

Answer: 10

(d) $\displaystyle\int_2^6 4\,[f(x)\cdot g(x) + f(x)]\,dx =$

Answer: Not possible

5. (8 points) Find the function $f(x)$ such that $f'(x) = 8x^3 - 10x + 6$ and $f(1) = -4$.

$f(x) = \int (8x^3 - 10x + 6)\,dx = 2x^4 - 5x^2 + 6x + C$

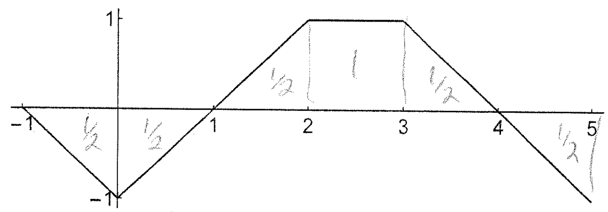
$f(1) = 2 - 5 + 6 + C = -4$

so $\quad C = -7$

Answer: $f(x) = 2x^4 - 5x^2 + 6x - 7$

Cont.

6. (8 points) The graph of a function f on the interval $[-1, 5]$ is given below



Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Compute the following values:

[Hint: Write out what g(x) is]

(a) $g(5) = \int_{-1}^{5} f(t)\,dt = \frac{1}{2}$

Answer: $\frac{1}{2}$

(b) $g(-1) = \int_{-1}^{-1} f(t)\,dt = 0$

Answer: 0

(c) $g'(1) = f(1) = 0$

Answer: 0

(d) $g'(2) = f(2) = 1$

Answer: 1

7. (4 points) Let h be the function defined by $h(x) = \int_{-x}^{x} f(t)\,dt$. Find $h'(x)$.

[Hint: Note the bounds of integration.]

$h(x) = \int_{-x}^{x} f(t)\,dt = \int_{0}^{x} f(t)\,dt - \int_{0}^{-x} f(t)\,dt$

Answer: $f(x) + f(-x)$

$h'(x) = f(x) - (-1)f(-x)$

Cont.

8. (26 points) Compute the following indefinite integrals.

(a) $\int \left[\frac{1}{\sqrt{x}} + x^{-4} + 2\cos x - 9\right] dx$

$\int (x^{-\frac{1}{2}} + x^{-4} + 2\cos(x) - 9)\,dx$

$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{-3}}{-3} + 2\sin(x) - 9x + C$

Answer: $2x^{\frac{1}{2}} - \frac{1}{3x^3} + 2\sin(x) - 9x + C$

(b) $\int 5(x^4 + 2x)(x^5 + 5x^2 + 7)^6\, dx$

Let $u = x^5 + 5x^2 + 7$ then $du = (5x^4 + 10x)dx$

So $\int 5(x^4+2x)(x^5+5x^2+7)^6 dx$

$= \int u^6 du = \frac{1}{7}u^7 + C$

$= \frac{1}{7}(x^5+5x^2+7)^7 + C$

Answer: $\frac{1}{7}(x^5+5x^2+7)^7 + C$

(c) $\int (\sin x)\sqrt{3 + 4\cos x}\, dx$

Let $u = 3 + 4\cos(x)$ then $du = -4\sin(x)\,dx$

So $\int \sin(x)\sqrt{3+4\cos(x)}\,dx = -\int \frac{1}{4}\sqrt{u}\,du$

$= -\frac{1}{4}\int u^{\frac{1}{2}} du = -\frac{1}{4}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= -\frac{1}{6}u^{3/2} + C$

Answer: $-\frac{1}{6}u^{3/2} + C = -\frac{1}{6}(3+4\cos(x))^{3/2} + C$

(d) $\int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right) dx$

Let $u = \frac{1}{x^2}$ then $du = -2x^{-3}dx$

then $\int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx = -\frac{1}{2}\int \sin(u)\,du$

$= \frac{1}{2}\cos(u) + C$

$= \frac{1}{2}\cos\left(\frac{1}{x^2}\right) + C$

Answer: $\frac{1}{2}\cos\left(\frac{1}{x^2}\right) + C$

9. (12 points) Compute the following definite integrals.

(a) $\int_1^3 (4x+2)\,dx$ $= 2x^2 + 2x \Big|_1^3$

$= 2\left((3^2-1^2)+(3-1)\right)$

$= 2(8+2) = 20$

Answer: 20

(b) $\int_0^{\pi/2} \sin^7(x)\cos(x)\,dx$ Let $u = \sin(x)$, then $du = \cos(x)dx$

when $x=0$, $u=0$; when $x=\frac{\pi}{2}$, $u=1$.

So $\int_0^{\frac{\pi}{2}} \sin^7(x)\cos(x)dx = \int_{u=0}^{u=1} u^7 du$

$= \frac{1}{8}u^8\Big|_0^1 = \frac{1}{8}$

Answer: $\frac{1}{8}$

10. (8 points) A particle is moving in a straight line with velocity $v(t) = 2t + 3\sqrt{t}$. Find the average velocity of the particle between time $t = 1$ and time $t = 9$.

$\text{Average} = \frac{1}{9-1}\int_1^9 (2t + 3\sqrt{t})\,dt$

$= \frac{1}{8}\int_1^9 (2t + 3t^{\frac{1}{2}})\,dt$

$= \frac{1}{8}\left[t^2 + 2t^{\frac{3}{2}}\right]_1^9$

$= \frac{1}{8}\left((9^2-1^2) + 2(9^{\frac{3}{2}} - 1^{\frac{3}{2}})\right)$

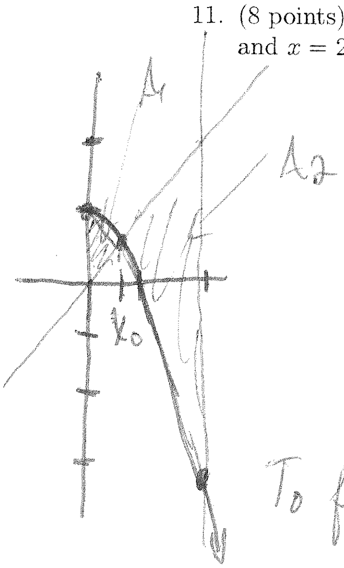
$= \frac{1}{8}\left[(81-1) + 2(27-1)\right]$

$= \frac{1}{8}\left[80+52\right] = \frac{1}{4}(40+26)$

$= \frac{1}{2}(20+13) = \frac{33}{2}$

Answer: $\frac{33}{2}$

11. (8 points) Find the area of the region bounded by the curves $y = 1 - x^2$ and $y = x$ and the lines $x = 0$ and $x = 2$.



So $\text{Area} = A_1 + A_2 = \int_0^{x_0}\left[(1-x^2) - x\right]dx + \int_{x_0}^{2}\left[x - (1-x^2)\right]dx$

$= x - \frac{1}{3}x^3 - \frac{1}{2}x^2\Big|_0^{x_0} + \frac{1}{2}x^2 - x + \frac{1}{3}x^3\Big|_{x_0}^{2}$

$= x_0 - \frac{1}{3}x_0^3 - \frac{1}{2}x_0^2 + \frac{1}{2}(2)^2 - 2 + \frac{1}{3}(2)^3 - \frac{1}{2}x_0^2 + x_0 - \frac{1}{3}x_0^3$

$= 2x_0 - x_0^2 - \frac{2}{3}x_0^3 + 2 - 2 + \frac{8}{3} = -1+\sqrt{5} - \frac{1}{2}(3-\sqrt{5}) - \frac{2}{3}(-2+\sqrt{5}) + \frac{8}{3}$

$= -1 - \frac{1}{2}(3) - \frac{2}{3}(-2) + \frac{8}{3} + \sqrt{5}\left(1 + \frac{1}{2} - \frac{2}{3}\right) =$

Answer: $\boxed{\frac{3}{2} + \frac{5}{6}\sqrt{5}}$

To find x_0 we need to solve

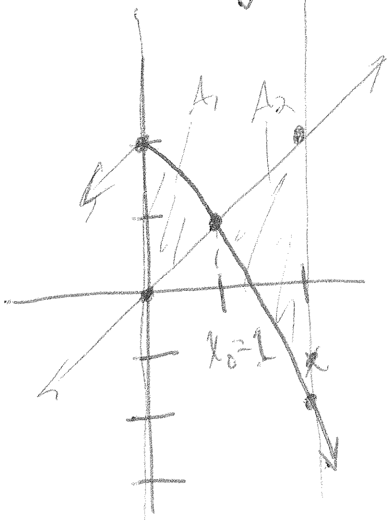
$1 - x^2 = x$ or $x^2 + x - 1 = 0$

$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$ So $x_0 = \frac{-1+\sqrt{5}}{2}$

$x_0^2 = \frac{1}{4}(6 - 2\sqrt{5}) = \frac{1}{2}(3-\sqrt{5})$ $x_0^3 = \frac{1}{4}(-3+4\sqrt{5}-5 = -2+\sqrt{5}$

The End.

With $y=2-x^2$ $\quad y=x$ $\quad x=0$ $\quad x=2$



$$\int_0^1 [(2-x^2)-x]\,dx + \int_1^2 [x-(2-x^2)]\,dx$$

$$= 2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \Big|_0^1 + \left[-2x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right]_1^2$$

$$= 2 - \tfrac{1}{2} - \tfrac{1}{3} + \left[-4 + 2 + \tfrac{8}{3} - \left(-2 + \tfrac{1}{2} + \tfrac{1}{3}\right)\right]$$

$$= \cancel{2} - \tfrac{1}{2} - \tfrac{1}{3} - \cancel{4} + \cancel{2} + \tfrac{8}{3} + 2 - \tfrac{1}{2} - \tfrac{1}{3}$$

$$= 2 - 1 - \tfrac{2}{3} + \tfrac{8}{3} = 1 + 2 = \boxed{3}$$

Interesting note, if you took the intersection to be $x=1$ for $y=1-x^2$; $y=x$ and calculated

$$\int_0^1 [(1-x^2)-x]\,dx + \int_1^2 [x-(1-x^2)]\,dx = 3$$ the same value.