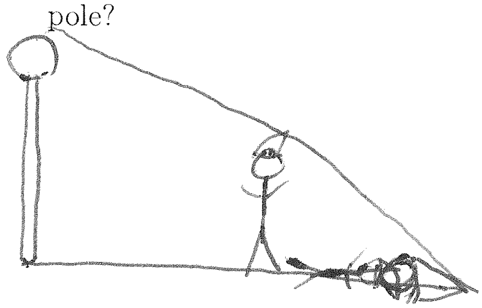
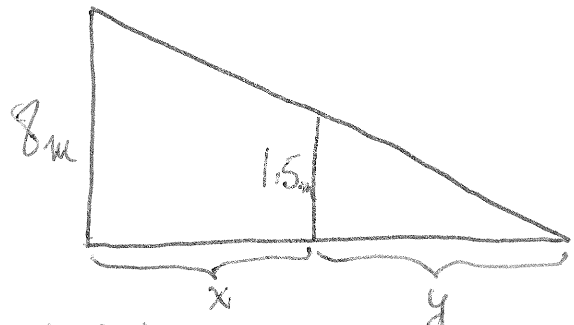
A boy and his shadow

Peter Pan is playing around in the streets of London late at night, while all the silly adults are asleep. Peter never wants to grow up, but is proud to say he is 1.5 meters tall. Peter lands at the base of an 8 meter tall light pole. When he looks down, he notices that his shadow has vanished, that rascal. Peter sets off from the base of the pole at a rate of 2 meters per second walking straight down the sidewalk and away from the pole in search of his shadow. After some time Peter notices his shadow has come back, and now it's growing! What fun! How quickly is Peter's shadow growing when Peter is 5 meters from the light pole?





Let x be the distance of Peter from the pole.
Let y be the distance of Peter's shadow from Peter.
By similar triangles we know:

$$\frac{8m}{1.5m} = \frac{x+y}{y} \Rightarrow 8y = 1.5x + 1.5y \Rightarrow y = \frac{3}{13}x.$$

The tip of the shadow moves by $\frac{d}{dt}(x+y)$ we want $\frac{dy}{dt}$.

$$\frac{d}{dt}(x+y) = \frac{d}{dt}\left(x+\frac{3}{13}x\right) = \frac{16}{13}\frac{dx}{dt} = \frac{16}{13}\cdot 2\,m/s = \frac{32}{13}\,m/s$$

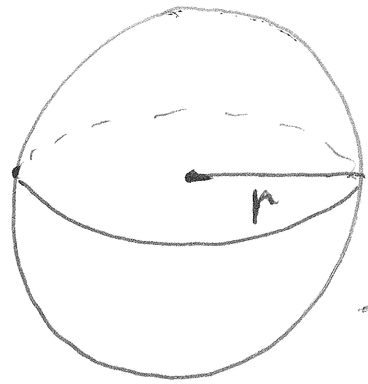
Then $\frac{d}{dt}(x+y) - \frac{dx}{dt} = \frac{dx}{dt} + \frac{dy}{dt} - \frac{dx}{dt} = \frac{dy}{dt}$ or

$$\frac{dy}{dt} = \frac{32}{13}\,m/s - 2\,m/s = \frac{6}{13}\,m/2$$

So Peter's shadow is growing at $\frac{6}{13}$ meters per second.

Frosty's Fate

Frosty the snow man is sweating. He's not just sweating, he's melting! He led the children down the streets of town and past the traffic cop, but now the sun's come out and Frosty's time on this Earth is limited. Frosty's body is made of three spheres of snow. His head is 30 cm across, his torso is 40 cm across, and his bottom is 50 cm across. Frosty is sweating away 100 cm^3 of snow per minute from his head. How quickly is Frosty's head shrinking, in terms of the radius of his head, when his head is 20 cm across? diameter



Let V be the volume of Frosty's head.
Let r be the radius.

Then $V = \frac{4}{3}\pi r^3$ so $\frac{dV}{dr} = 4\pi r^2$.

By chain rule $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$$\frac{-100\text{ cm}^3}{\text{min}} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-100\text{ cm}^3}{4\pi r^2\text{ min}}$$

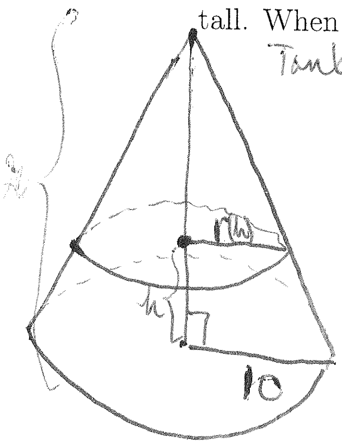
$$\left.\frac{dr}{dt}\right|_{r=10} = \frac{-100\text{ cm}^3}{4\pi(10\text{cm})^2\text{ min}} = -\frac{1}{4\pi}\text{ cm/min}.$$

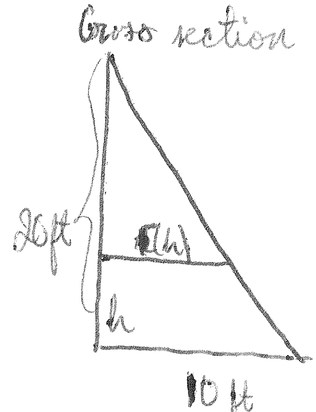
The radius of Frosty's head is decreasing at a rate of $\frac{1}{4\pi}$ centimeters per minute.

Real solution to original problem.

No Mr. Bond, I expect you to die

James has been in these types of situations before, but this attempt just seems kind of lazy on Goldfinger's part. James Bond is trapped in a right conical tank that is filling with water from the apex above his head. The tank appears to be about 20 feet tall with a base that is 20 feet in diameter, and he estimates that the tank is filling at a rate of 200 cubic feet per minute (He's been in this situation before). James knows he has about 10 and a half minutes until the tank fills entirely, but James isn't 20 feet tall. James is about 6 feet tall. When the water reaches James's shoulders at 5.5 feet, how quickly is the water rising?





Let V_{tip} be the volume of air in the top part of the tank above the water. Let V be the volume of the water in the tank. Let $r(h)$ be the radius of the surface of the water at height h in the tank.

By similar triangles $\frac{20-h}{20} = \frac{r(h)}{10} \Rightarrow r(h) = 10 - \frac{1}{2}h$

The volume of the entire tank is $\frac{1}{3}\pi(10)^2 \cdot 20 = \frac{1}{3}\pi(2000)$ cubic feet.

$V_{tip} = \frac{1}{3}\pi(r(h))^2 \cdot (20-h) = \frac{1}{3}\pi\left(10-\frac{1}{2}h\right)^2 \cdot (20-h)$

$V = \frac{1}{3}\pi(2000) - \frac{1}{3}\pi\left(10-\frac{1}{2}h\right)^2 \cdot (20-h) = \frac{1}{3}\pi\left(2000 - \left(10-\frac{1}{2}h\right)^2(20-h)\right)$

Then $\frac{dV}{dh} = \frac{1}{3}\pi\left((-2)\left(10-\frac{1}{2}h\right)\left(-\frac{1}{2}\right)(20-h) + (-1)\left(-\left(10-\frac{1}{2}h\right)^2\right)\right)\ ft^2$

$\left.\frac{dV}{dh}\right|_{h=5.5} \approx 165.12996\ldots\ ft^2$

By chain rule then $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ or $\frac{dh}{dt} = \frac{dV}{dt} \cdot \left(\frac{dV}{dh}\right)^{-1}$

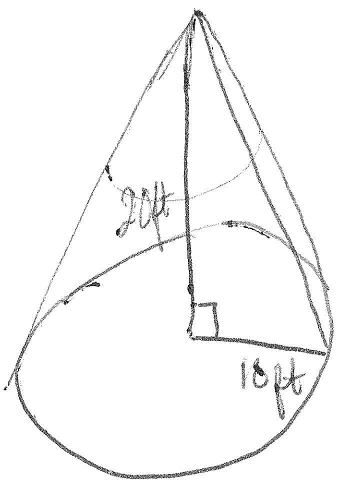
Then $\left.\frac{dh}{dt}\right|_{h=5.5} = \frac{200 ft^3}{min} \cdot \frac{1}{165.12996 ft^2} \approx 1.21$ ft per minute

The water is rising at a rate of 1.21 feet per minute then.

Changed to easier problem.

No Mr. Bond, I expect you to die

James has been in these types of situations before, but this attempt just seems kind of lazy on Goldfinger's part. James Bond is trapped in a right conical tank that is filling with ~~water~~ Sand from the apex above his head. The tank appears to be about 20 feet tall with a base that is 20 feet in diameter, and he estimates that the tank is filling at a rate of 200 cubic feet per minute (He's been in this situation before). James knows he has about 10 and a half minutes until the tank fills entirely, but James isn't 20 feet tall. James is about 6 feet tall. When the ~~water~~ sand reaches James's shoulders at 5.5 feet, how quickly is the ~~water~~ sand rising?



Let V be volume & h be height of ~~water.~~ Sand.

$$V = \frac{h}{3}\pi r^2 = \frac{h}{3}\pi(100\,ft^2)$$

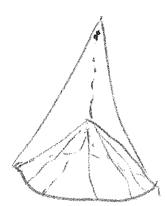
$$\frac{dV}{dh} = \frac{100}{3}\pi\,ft^2$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$\frac{200\,ft^3}{min} = \frac{100}{3}\pi\,ft^2\,\frac{dh}{dt}$$

$$\frac{dh}{dt} = \cancel{\frac{2}{3\pi}\frac{ft}{min}} \quad \frac{6}{\pi}\frac{ft}{min}$$

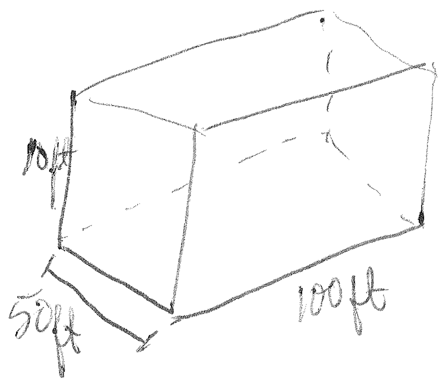
Idea is that sand fills like a cone... water does not.



The walls are moving

Princess Leia Organa had accepted that she was probably going to die, and after what happened to Alderaan she had lost all hope. Then some scruffy-looking nerf herders came along and broke her out of her cell, but they have no escape plan. Now, after jumping down the garbage chute, it appears that instead of being executed or struck down by blaster fire, she is now to be crushed in the garbage masher with her would be rescuers. The garbage masher they find themselves in is 10 feet tall, 100 feet long, and 50 feet wide. If the masher brings the walls in at a rate of 6 feet per minute, subtracting from the 50 foot width, then what will be the change in volume over time when the C3PO manages to halt the compacter with only 4 feet left until they would be mashed?

Let V be the volume; w the width.



$$V = l \cdot w \cdot h$$

$$V = 100\,ft \cdot 10\,ft \cdot w$$

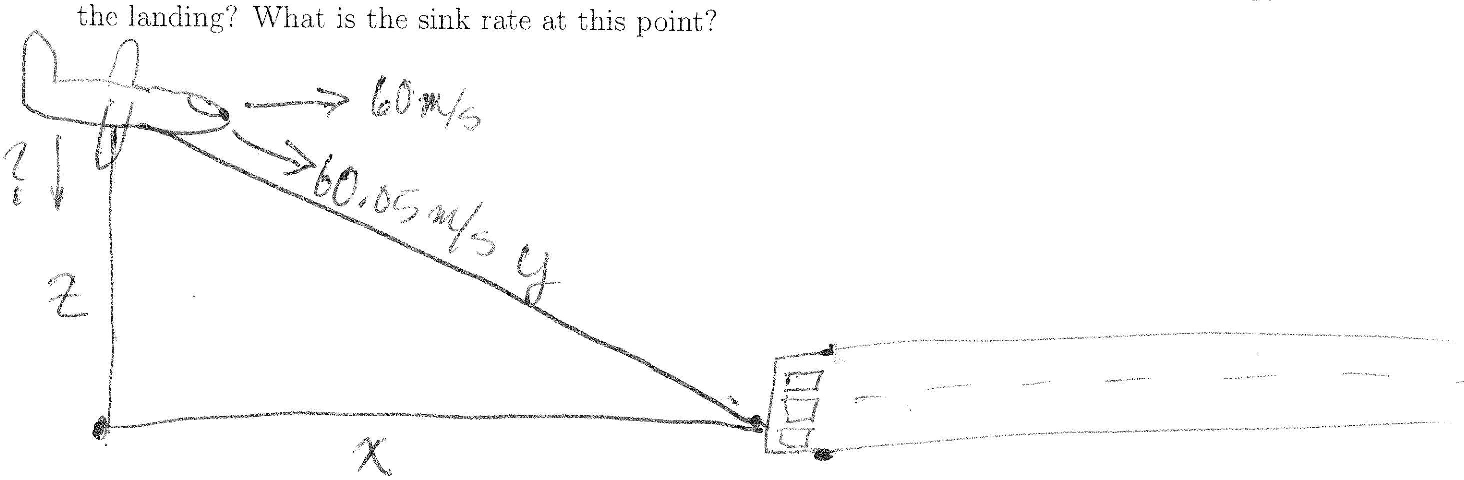
$$\frac{dV}{dw} = 1000\,ft^2$$

$$\frac{dV}{dt} = \frac{dV}{dw} \cdot \frac{dw}{dt} = 1000\,ft^2 \cdot \frac{-6\,ft}{min} = \frac{-6000\,ft^3}{min}$$

The volume of the masher is decreasing by $\frac{6000\,ft^3}{min}$ or $100\,ft^3$ per second.

The fate of Flight 209

Flight 209 from LAX to ORD is already in trouble. Both pilots have taken ill, and now a nervous Lt. Striker must take up the controls despite a crippling fear of flying. However, Dr. Rumak has more bad news for our brave pilot: "It appears the altimeter has stopped working," Rumak informs Striker. "Surely you can't be serious," Striker exclaims. "I am serious, and don't call me Shirley," replies Rumak. Striker knows the following information. The plane is headed for the runway at the correct approach speed of 60 meters per second. He also knows the distance from a beacon at the end of the runway is decreasing at a rate of 60.05m per second. When he flies over the outer beacon, which is 1000m from the end of the runway, he will be about 1004m from the beacon at the end of the runway, and so at the minimum approach height of 90m. At this point he must decide whether to land or miss the approach. If his sink rate is over 2m per second, then he should abort. Will Striker abort the landing? What is the sink rate at this point?



Let x, y, z be as labeled. Then $\frac{dx}{dt} = \frac{-60m}{sec}$ $\frac{dy}{dt} = \frac{-60.05m}{sec}$

$$x^2 + z^2 = y^2$$

So $\frac{d}{dt}(x^2+z^2) = \frac{d}{dt}(y^2)$

$$2x\frac{dx}{dt} + 2z\frac{dz}{dt} = 2y\frac{dy}{dt}$$

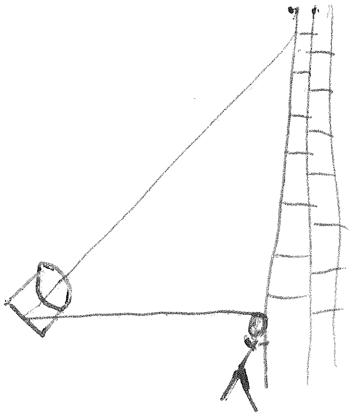
or $\frac{dz}{dt} = \frac{1}{z}\left(y\frac{dy}{dt} - x\frac{dx}{dt}\right)$ So $\frac{dz}{dt}\Big|_{(1000,1004,90)} = \frac{1}{90}\left(1004\left(\frac{-60.05m}{sec}\right) - 1000\left(\frac{-60m}{sec}\right)\right)$

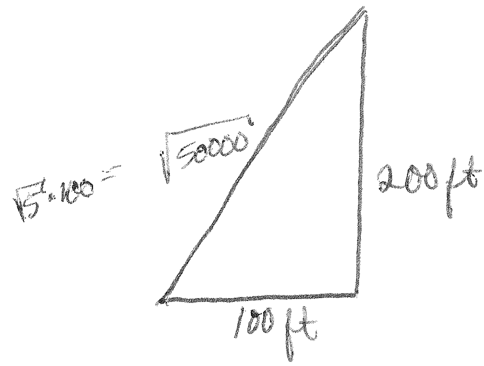
$= -3.22 \, m/sec$

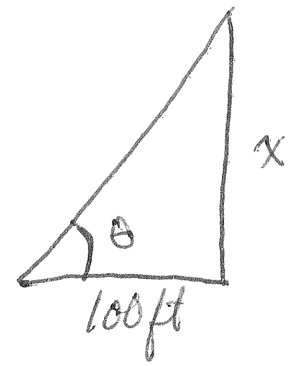
Abort! The sink rate is 3.22 m/sec.

Sprint to freedom

Dr. Richard Kimble, falsely accused of his wife's murder, has just escaped custody and is on the run. He is attempting to dodge Marshall Samuel Gerard who has set up roadblocks and spotlights. He has no choice but to run along an exposed wall that is 100 feet away from a spotlight at its nearest point. The spotlight rotates at 1 revolution per minute. Dr. Kimble can sprint 35 feet per second, if he must out-sprint the speed of the spotlight 200ft along from its nearest point to the wall, will he make it? What is the speed of the spotlight 200ft along from the nearest point to the wall?







Let x be the position of the light on the wall from the nearest point an θ its angle from that point.

Then $\tan(\theta) = \frac{x}{100ft}$

$$\frac{d}{dt}\left(\tan(\theta)\right) = \frac{d}{dt}\left(\frac{x}{100ft}\right) = \frac{dx}{dt}\,\frac{1}{100ft}$$

$$\frac{d\theta}{dt}\sec^2(\theta) = \frac{dx}{dt}\cdot\frac{1}{100ft}$$

$$\frac{2\pi}{min}\cdot\left(\frac{\sqrt{5}\cdot 100}{100}\right)^2 = \frac{dx}{dt}\,\frac{1}{100ft}$$

$$\frac{dx}{dt} \approx 52\ ft/sec$$

Dr Kimble loses. The spotlight is going 52 ft/sec.