Solution to 12.1

Suppose Q is a point on ℓ but not on π. Then there is a unique line (projection) through Q perpendicular to π. Call the point of intersection of π with this line Q'. Then since PF and QQ' are perpendicular to the same plane, they are parallel. Call the plane they live in π'. In π' $PFQ'Q$ forms a quadrilateral. By our assumption on ℓ $\angle QFP = 90°$, but $Q'F$ is a line in π and so $\angle PFQ'$ is 90 degrees as well. This is a contradiction since Q is interior to $\angle PFQ'$ (unless Q is Q'. Thus ℓ is in π.

Solution to 12.2

Clearly π has more than one point. Let π' be the plane through A perpendicular to the line AB. We want to show $\pi' = \pi$. Let C be in π' then by the definition of orthogonality to the plane $AC \perp AB$ so C is in π. Let C be in π, then the line CA perpendicular to the line BA and so by previous problem CA lies in π' and so C is in π'. So by double inclusion $\pi = \pi'$.

Solution to 12.12

Let ℓ_1 and ℓ_2 be parallel in the plane π_2. Let π_1 be another plane not orthogonal to π_2. Take the orthogonal projections of the lines onto the plane π_1, call the projected lines p_1 and p_2. Suppose p_1 and p_2 are not parallel. Then they intersect at a point, A say. Now consider the planes $\omega_1 = \pi(\ell_1, p_1)$ and $\omega_2 = \pi(\ell_2, p_2)$. Notice that since our projections are orthogonal, the dihedral angles between ω_1, ω_2 and π_1 are 90°. Since p_1 and p_2 intersect and are in ω_1 and ω_2 the two planes must also intersect in a line ℓ_3 orthogonal to π_1 at A. But since ℓ_3 is the orthogonal line through $A \in p_1 \cap p_2$ it must intersect both ℓ_1 and ℓ_2 at the points B and C say. If $B = C$ then we have a contradiction to our original lines being parallel. If B and C are distinct in π_2 then ℓ_3 is in π_2 and orthogonal to π_1 and so we contradict our assumption of π_1 not being orthogonal to π_2. $\square$

Solution to 12.13

Consider the triangles formed by the extended lines of the segments and their projections. That is, let $\ell_1 = \overleftrightarrow{AB}$ and $\ell_2 = \overleftrightarrow{CD}$. Let p_1 and p_2 be their respective projections. Call the respective projected points A', B', C', and D', and call the intersection of ℓ_1 and p_1 O_1 and the intersection of ℓ_2 and p_2 O_2 With the additional assumption given to you, you know that $\angle AO_1A' \cong \angle CO_2C'$. This holds in general. So then $\triangle BO_1B' \sim \triangle DO_2D' \sim \triangle AO_1A' \sim \triangle CO_2C'$. For the simple case, we can assume $A = A' = O_1$ and $C = C' = O_2$. Then the result follows directly. For the general case, apply the ratios of the sides of the triangle twice.