Math 45021	Euclidean Geometry	Spring 2017
April 26, 2017	Exam 2	Matt Alexander

Name: Key Score: ______ /50

> Please follow the directions provided in each section. You may assume that the parallel postulate holds for all questions on this exam.

True or False (1 pt each): *For each of the following circle T if the statement is true or F if the statement is false.*

1. T F All vertices of a triangle lie on its circumcircle.

2. T F Any quadrilateral can be inscribed by a cirlce with all vertices on the circle.
   Defined by 3 points

3. T F The centroid of a triangle is the point of intersection of the three medians.

4. T F The product of the exradii and inradius is equal to the circumradius.
   $\leftarrow K^2$

5. T F In an equilateral triangle, the incenter, circumcenter, and orthocenter are the same.

6. T F It is possible for two pairs of angle bisectors of a triangle to meet at different points.
   Incenter is intersection of all 3

7. T F Rotation around a point preserves distances between points.

8. T F The reflection of a triangle over a line results in a triangle of half the area.
   Same area

9. T F There is one unique plane containing any two points in three dimensions.
   need 3

10. T F There is one unique plane containing any two intersecting lines in three dimensions.

11. T F There exists a polytope in three dimensions with 8 vertices, 12 edges, and 6 faces.

12. T F If two polygons with vertices on integer lattice points have the same number of interior points and boundary points on the integer lattice, then they are equidecomposible.

Multiple Choice (2 pts each): *Circle the letter of the best response.*

13. The set of all points equidistant from the endpoints of a segment form the
    (a) Angle bisector
    (b) Median
    (c) Perpendicular bisector
    (d) Altitude

14. Which of the following is true of the complementary triangle $\triangle DEF$ of $\triangle ABC$
    (a) The triangles are similar with ratio 2.
    (b) The triangles are congruent.
    (c) The triangles have the same circumcircle.
    (d) $\triangle ABC$ and $\triangle DEF$ have the same incenter.

15. The intersection of the perpendicular bisectors of a triangle is the
    (a) Orthocenter
    (b) Incenter
    (c) Circumcenter
    (d) Centroid

16. Given a polygon in the plane with vertices on integer points such that there are 20 lattice points in the interior and 6 on the boundary, what is the area of the polygon?
    (a) 15
    (b) 16
    (c) 23
    (d) 22

$20 + \frac{1}{2}(6) - 1$

22

Short answers (2 pt each): *Find the indicated values.*

17. Given $a = 2$, $b = 3$, $c = 4$, and the area of the triangle is 10 (approx). Find the circumradius R.

$abc = 4RK$

$\cancel{2} \cdot 3 \cdot \cancel{4} = \cancel{4} R \cdot \cancel{10}^{5}$

$R = \frac{3}{5}$

Answer: $R = \frac{3}{5}$

18. Given $r_a = 3$, $r_b = 4$, and $r_c = 5$. Find the inradius r.

$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$

$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{1}{r}$

$\frac{20}{60} + \frac{15}{60} + \frac{12}{60} = \frac{47}{60} = \frac{1}{r}$

Answer: $r = \frac{60}{47}$

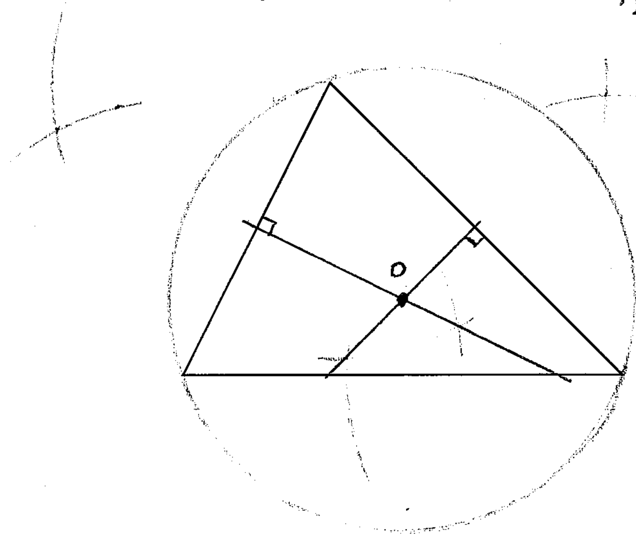
Cont.

Constructions (3 pts each): **Pick 2 of 3**

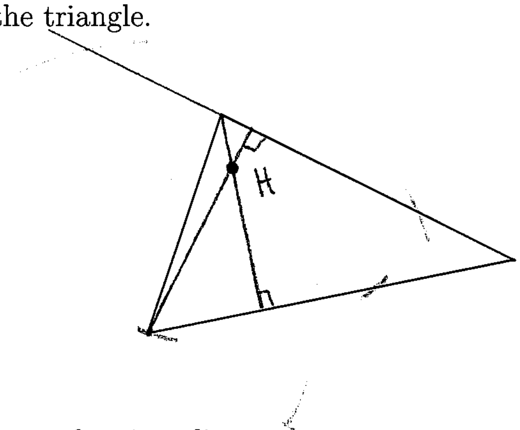
Create the drawings with a compass and straightedge. Show each step clearly.

19. Construct the circumcircle of the triangle.

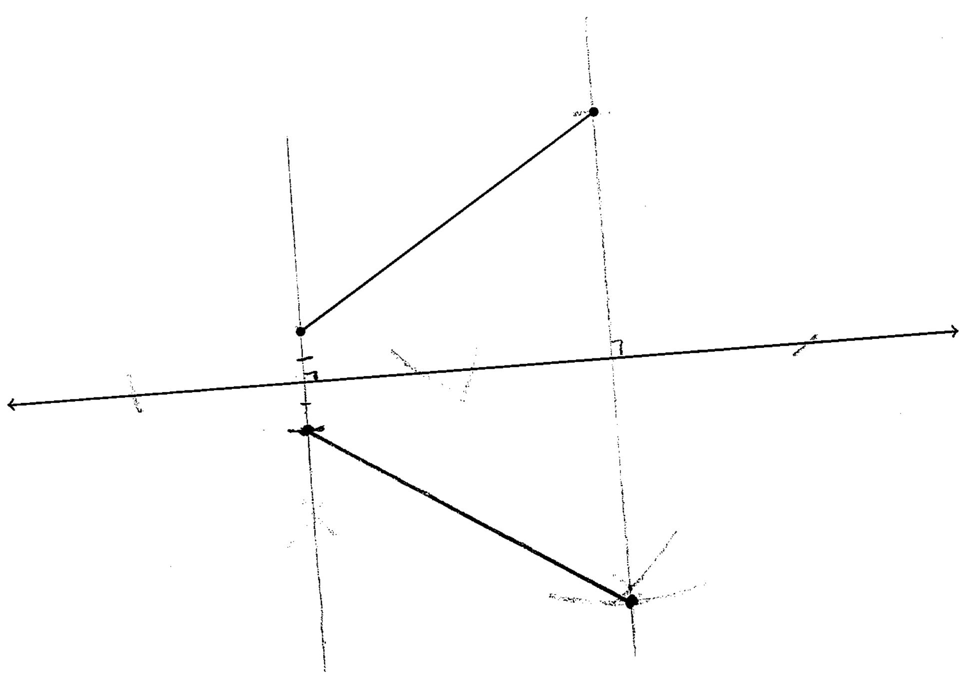
1) Find the circumcenter which is the intersection of all perp bisectors
2) Draw circle



20. Find the orthocenter of the triangle.



21. Reflect the line segment over the given line.

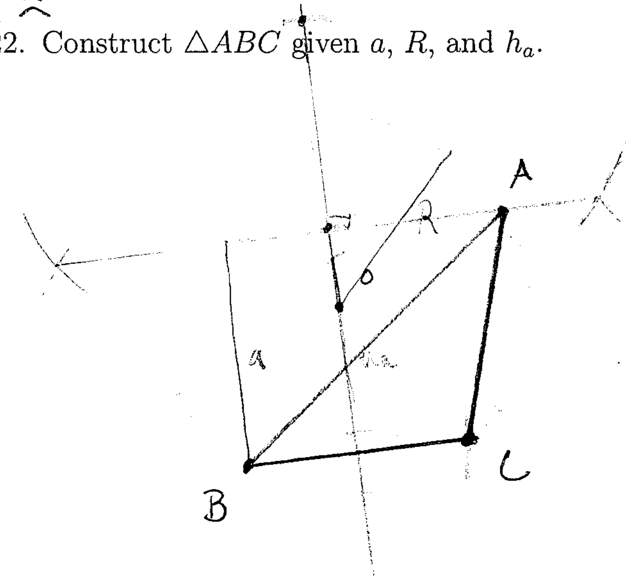


Indirect Constructions (3 pts each): Pick 2 of 3

Create the drawings with a compass and straightedge. Write and label each step clearly.

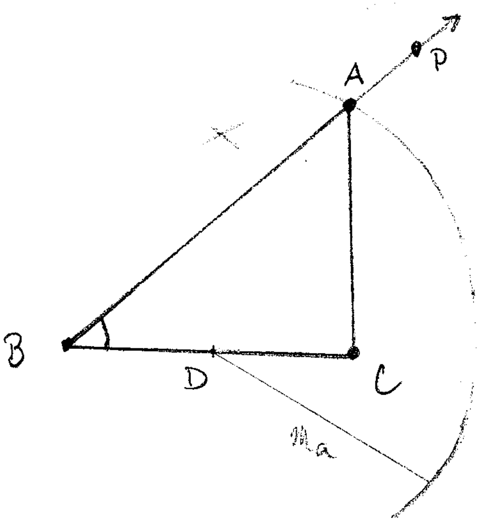
Compass broke :(

22. Construct $\triangle ABC$ given a, R, and h_a.



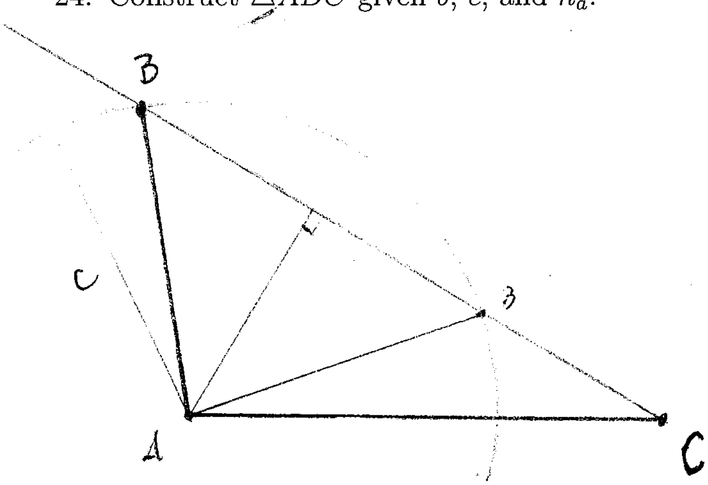
1) Draw the circumcircle of radius R center O
2) Choose B on the circle
3) Draw circle center B radius a to intersect ① and find C
4) Draw a line parallel to BC at a distance h_a.
5) Label A the intersection of ① & ④

23. Construct $\triangle ABC$ given a, $\angle B$, and m_a.



1) Draw segment $\overline{BC}$ of length a.
2) Copy the angle $\angle B$ on $\overline{BC}$ with ray $\overrightarrow{BP}$
3) Find the midpoint of $\overline{BC}$, label D
4) Draw a circle center D radius m_a
5) Label the intersection of ② & ④ A

24. Construct $\triangle ABC$ given b, c, and h_a.

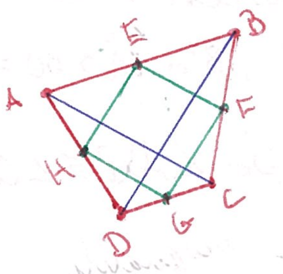


1) Draw segment AC with length b
2) Draw a circle center A of radius c
3) Draw a circle center A of radius h_a
4) Draw the line tangent to ③ ~~through~~ C
5) Label the intersection of ④ & ② B

Cont.

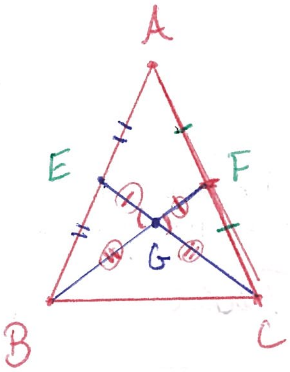
Proofs (5 pts each): Choose 3 of 5 *Justify the following statements with a proof.*

25. Let $ABCD$ be any quadrilateral. Let the midpoints of the sides be E, F, G, and H. Prove that $EFGH$ is a parallelogram.



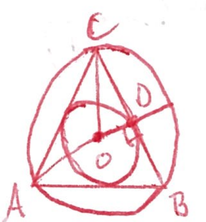
Consider in $\triangle ABD$ $\overline{HE}$ is a midsegment and so $\overleftrightarrow{HE} \parallel \overleftrightarrow{DB}$. If you wanted to prove this then show $\triangle AHE \sim \triangle ADB$.
Now in $\triangle CDB$ $\overline{GF}$ is a midsegment so $\overleftrightarrow{DB} \parallel \overleftrightarrow{GF}$. Thus $\overleftrightarrow{GF} \parallel \overleftrightarrow{HE}$
By a similar argument with $\triangle ACB$ & $\triangle ACD$ $\overleftrightarrow{EF} \parallel \overleftrightarrow{HG}$.
Thus $EFGH$ is a parallelogram. $\square$

26. Let $\triangle ABC$ be such that $m_b = m_c$. Prove $\triangle ABC$ is isosceles.



Let E be the midpoint of $\overline{AB}$ & F the midpoint of $\overline{AC}$
Then $AE = EB$ and $AF = FC$. We are given $BF = m_b = m_c = CE$.
The intersection of BF and CE is the centroid G. We know that $\triangle EFG \sim \triangle CGB$ with ratio $\frac{2}{3}$. Thus $\overline{EG} \cong \overline{GF}$ and $\overline{BG} \cong \overline{GC}$. Which, along with $\angle EGB \cong \angle FGC$ by vertical angles gives us $\triangle EGB \cong \triangle FGC$ by SAS. So $EB = FC$ which gives $AB = 2EB = 2FC = AC$. Thus $\triangle ABC$ is isosceles. $\square$

27. Prove that in an equilateral triangle the circumradius is twice the inradius. That is, $R = 2r$.

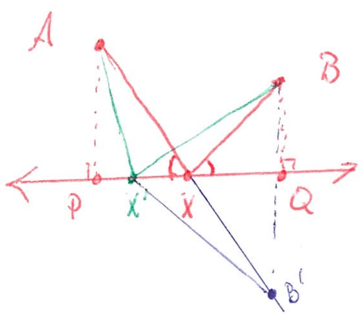


$\boxed{\text{2 Proofs}}$ Cheap way via Euler's Theorem $OI^2 = R(R-2r)$
The distance from the circumcenter to the incenter is 0 in an equilateral triangle and so either $R = 0$ (impossible) or $R = 2r$. $\square$

Better proof Since $G = O = I$ the line AO is a median and a perpendicular bisector meeting at D on side $\overline{BC}$. OD is the inradius and AO is the circumradius. The centroid splits the median in a 1:2 ratio thus $AO = 2\,OD$ or $R = 2r$. $\square$

Cont.

28. Let A and B be two points on one side of a line ℓ. Let P and Q be the projections of A and B onto the line ℓ. Let X be the point on the line ℓ between P and Q such that $\angle AXP \cong \angle BXQ$. Show that X is the point that minimizes the path $A - X - B$. [Hint: You may assume that a straight line has the shortest distance between any two points.]



Let X' be any other point on ℓ between P & Q and B' be the reflection of B over ℓ (i.e. B' on $\overleftrightarrow{BQ} \ni BQ = B'Q$)

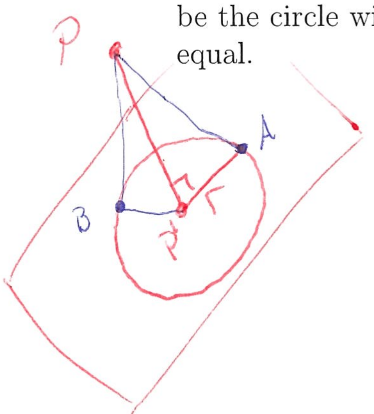
Then $\angle QXB' = \angle QXB$ by reflection. So $\angle AXP \cong \angle QXB'$.

Hence $\angle AXB' = 180°$. So by triangle inequality for $\triangle AX'B'$ $AX' + X'B' > AB' = AX + XB'$

But $XB' = XB$ so $AX' + X'B' > AX + XB$ for any $X' \neq X$

Hence A-X-B is the ~~shortest~~ path. ∎

29. Let π be a plane and P be a point not on π. Call P' the orthogonal projection of P onto π. Let C be the circle with center P' and radius r in π. Show that the distance from P to any point on C is equal.



Let A and B be two points on C. Then $\overline{P'A} \cong \overline{P'B}$ are radii of the circle $\overline{P'P} \cong \overline{PP'}$ and $\angle PP'A \cong \angle PP'B$. So by SAS $\triangle PP'B \cong \triangle PP'A$.

Thus $\overline{PB} \cong \overline{PA}$ as desired for any points on C. ∎