RESEARCH STATEMENT

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1. Introduction

The main topics of my research interests are Convex and Discrete Geometry. I am interested in finding versions of classical facts from Convex Geometry and Geometric Tomography in discrete and non-linear settings. My interest in this area began during my time as an undergraduate in a summer REU where we explored a discrete version of Aleksandrov's uniqueness theorem. Our exploration followed a call by Gardner, Gronchi, and Zong in [GGZ] to begin bringing the theory of discrete tomography in line with the fuller theory of continuous geometric tomography.

Geometric Tomography concerns the reconstruction of objects from incomplete data such as projections or sections. The objects of interest are often taken to be convex bodies so that the tools of convex geometry may be applied. There is a rich theory for continuous convex bodies in $\mathbb{R}^n$, however, much less has been done for discrete convex lattice sets in $\mathbb{Z}^n$. Here, a convex body is a compact convex set with non-empty interior, and a convex lattice set is a set of points equal to the intersection of $\mathbb{Z}^n$ with a convex body. A large reason for this current disparity is that many of the basic results for continuous convex bodies fail in the discrete setting. For example, for two convex bodies in $\mathbb{R}^n$ their Minkowski sum is convex. However, in the discrete case this does not remain true. Another example is Brunn's theorem which says that for any origin-symmetric convex body and given any unit vector the section of largest volume perpendicular to the unit vector passes through the origin. We can formulate this theorem by taking volume to be the cardinality of a discrete set, but we again find that the theorem does not hold.

In [AHZ] we explored a discrete question that follows from questions relating to the isomorphic Busemann-Petty problem. In particular we found that for a convex origin symmetric body K the discrete volume of the largest slice of K is larger than the discrete volume of K up to a constant depending only on the dimension. We also found the best possible bound in the case of unconditional bodies still depends on the dimension, and we generalized the result to slices of higher codimension.

Another well known open problem is the conjecture of Mahler related to the extreme values of the volume product of a convex body. In [AFZ2] we investigated several cases for the class of polytopes with less than a certain number of vertices. In particular, we independently verified a result from [MR2] that the regular N-gon is the polygon of maximal volume product for all polygons with N vertices. We also study the maximal bodies in the class of convex polytopes with $n+2$ vertices, and symmetric poltyopes with $2n+4$ vertices. In [AFZ1] we explore a more discrete version of the volume product that comes from embedding the space of Lipschitz functions over a metrics space as a symmetric polytope with conditions on its vertices, called the Lipschitz-free space. We study the maximal body in this setting in dimension two, and the minimal body in dimension three.

2. Inequalities in Discrete Tomography

Let us denote by $\mathrm{vol}_n(K)$ the n-dimensional volume of a body $K \subset \mathbb{R}^n$. For a unit vector $\xi \in \mathbb{R}^n$ will denote by $\xi^\perp = \{\xi \in \mathbb{R}^n : \xi \cdot x = 0\}$ a hyperplane orthogonal to ξ. The famous Radon theorem tells us that a convex, symmetric body is uniquely defined by the function $\mathrm{vol}_{n-1}(K \cap \xi^\perp)$ (see [Ga]). The analogous question in the discrete settings was considered in [GGZ] and turned out to have a trivially affirmative answer. What we are interested in are inequalities comparing the volumes of convex bodies and inequalities involving the size of section of convex symmetric bodies. The original Busemann-Petty problem was posed in 1956 (see [BP]). Let K and L be the origin-symmetric convex bodies in $\mathbb{R}^n$ such that $\mathrm{vol}_{n-1}(K \cap \xi^\perp) \leq \mathrm{vol}_{n-1}(L \cap \xi^\perp)$ for every $\xi \in \mathbb{S}^{n-1}$. Does it necessarily follow that $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$? This problem was solved at the end of the 1990's; we refer to [Zh, GKS, Ko1] for the solution and historical details. The answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. The fact that the answer is negative in the high dimensional case naturally leads to the question if the situation can be saved with the help of an absolute constant. More precisely, does there exist a constant $C > 1$ such that for any $n \geq 2$ and any origin-symmetric convex bodies $K, L \subset \mathbb{R}^n$ such that $\mathrm{vol}_{n-1}(K \cap \xi^\perp) \leq \mathrm{vol}_{n-1}(L \cap \xi^\perp)$ for every $\xi \in \mathbb{S}^{n-1}$ we get $\mathrm{vol}_n(K) \leq C\mathrm{vol}_n(L)$? The above is called the Isomorphic Busemann-Petty problem and is considered one of the most exciting open problems in Convex Geometry. Using the F. John theorem (see [MS]) it is not hard to show that C can be chosen as small as $\sqrt{n}$, the best known estimate is due to Klartag [Kl], who proved that C can be chosen of the order of $n^{1/4}$ (which improves the previous estimate of Bourgain [Bo1, Bo2]). This problem is equivalent to the so called "slicing problem", which asks if a symmetric convex body of volume 1 must have a central slice of large enough volume. More precisely, does there exists an absolute constant $c > 0$ such that for any convex symmetric body $K \subset \mathbb{R}^n$ with $\mathrm{vol}_n(K) = 1$, $\max_{\xi \in \mathbb{S}^{n-1}} \mathrm{vol}_{n-1}(K \cap \xi^\perp) > c$? Again the best known constant follows from the work of Klartag and is of order $n^{-1/4}$.

We note that the discrete version of the Busemann-Petty problem has a trivially affirmative solution in all dimensions. Indeed, let K and L be the origin-symmetric convex bodies $\mathbb{R}^n$ such that $\#(K \cap \xi^\perp \cap \mathbb{Z}^n) \leq \#(L \cap \xi^\perp \cap \mathbb{Z}^n)$ for every $\xi \in \mathbb{S}^{n-1}$, where $\#K$ is the cardinality of the set. Then we can always select a hyperplane $\xi^\perp$ which intersects $\mathbb{Z}^n$ just by a line (i.e. a subspace of dimension 1) and thus the inequality on hyperplanes gives immediately an inequality on all lines through the origin, which results in the fact that $\#(K \cap \mathbb{Z}^n) \leq \#(L \cap \mathbb{Z}^n)$.

The situation is completely different for the slicing problem. It was noticed in [KoZ] that the equivalence of slicing problem to the Busemann-Petty problem requires the measure (volume) to be a homogeneous and the cardinally of $K \cap \mathbb{Z}^n$ is not a homogeneous measurement. Recently, Koldobsky [Ko2, Ko3, Ko4, Ko5, Ko6] developed an approach to the slicing problem for general measures via stability results and Zvavitch's solution of the Busemann-Petty problem for general measures [Z1, Z2]. Koldobsky was able to prove that if μ is a measure with a strictly positive even density on $\mathbb{R}^n$, then for any convex symmetric body $K \subset \mathbb{R}^n$

$$\mu(K) \leq Cn^{1/2} \max_{\xi \in \mathbb{S}^{n-1}} \mu(K \cap \xi^\perp) \, \mathrm{vol}_n(K)^{\frac{1}{n}}$$

where C is an absolute constant. It is an open question if the dependence on n can be improved in the above inequality. However, Koldobsky also proposed a discrete version of the slicing problem:

Problem: *Does there exists an absolute constant $C > 0$ such that for any convex symmetric body $K \subset \mathbb{R}^n$*

$$\#(K \cap \mathbb{Z}^n) \leq C \max_{\xi \in \mathbb{S}^{n-1}} \#(K \cap \xi^\perp \cap \mathbb{Z}^n) \ \mathrm{vol}_n(K)^{\frac{1}{n}}.$$

It is essential to note that the problem does not follow from the standard slicing inequality or slicing inequality for general measures. Both of these inequalities turn out to be trivial in dimension 2, but the discrete version is non-trivial even in two dimensional case. In dimension two one may obtain an estimate based on the well known Minkowski's First theorem and Pick's theorem. However, for higher dimensions, a discrete version of Brunn's theorem is needed. Thus we showed in [AHZ] the following theorems.

Theorem 1. *Consider a convex, origin-symmetric body $K \subset \mathbb{R}^d$ and a lattice $\Gamma \subset \mathbb{R}^d$ of rank d, then*

$$\#(K \cap \boldsymbol{\xi}^\perp \cap \Gamma) \geq 9^{-(d-1)} \#(K \cap (\boldsymbol{\xi}^\perp + t\boldsymbol{\xi}) \cap \Gamma), \text{ for all } t \in \mathbb{R} \text{ and } \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

From this discrete versions of Brunn's theorem, we were then able to answer the general question for maximal slices of any codimension.

Theorem 2. *Let $K \subset \mathbb{R}^d$ be an origin-symmetric convex body with $\dim(K \cap \mathbb{Z}^d) = d$. Then*

$$\#K \leq O(1)^d \, d^{d-m} \max\{\#(K \cap H) : H \in \mathrm{G}_{\mathbb{Z}}(m,d)\} \ \mathrm{vol}_d(K)^{\frac{d-m}{d}}. \tag{1}$$

Then, for $m = d-1$ we obtain the estimate for hyperplane slices

$$\#K \leq O(1)^d \max_{\boldsymbol{\xi} \in \mathbb{S}^{d-1}} \left(\#(K \cap \boldsymbol{\xi}^\perp)\right) \ \mathrm{vol}_d(K)^{\frac{1}{d}}. \tag{2}$$

It is not clear if this bound may be improved in general. However, for the special class of unconditional bodies, we are able to get the following sharper estimate whose equality comes from the cross-polytope.

Theorem 3. *Let $K \subset \mathbb{R}^d$ be an unconditional convex body with $\dim(K \cap \mathbb{Z}^d) = d$. Then*

$$\#K \leq O(d) \max_{i=1,\dots,d} \left(\#(K \cap \boldsymbol{e}_i^\perp)\right) \ \mathrm{vol}_d(K)^{\frac{1}{d}},$$

where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_d$ are the standard basis vectors in $\mathbb{R}^d$. Moreover, this bound is the best possible.

2.1. **Future Work.** There are many future directions possible for work in this area. First, as stated above, is studying whether it is possible to improve the bound in theorem 2. One likely area to begin improving is in the bound given in theorem 1. Another question is if the continuous volume of the body, needed to preserve homogeneity, can be eliminated from the estimate. There are also questions what can be said when restricting the size of the body to be either large or small. In [R] the constant is improved for bodies with volume less than C^{d^2}.

Further, there are many unanswered questions still open from [GGZ]. For example, the problem that I worked on as an undergraduate of a discrete version of Aleksandrov's uniqueness theorem is still open with recent results on related questions in [RYZ] about determining the convex lattice set uniquely via the surface area of projections. It would also be possible to examine similar questions regarding the surface area of slices.

3. Special Cases for Volume Product

Let K be convex body in $\mathbb{R}^n$, symmetric with respect to the origin. One of the hardest *open* problems in convex geometry is to understand the connection between the *volumes* of K and K^* (the polar body of K, $K^* := \{x \in \mathbb{R}^n : x \cdot y \leq 1 \, \forall y \in K\}$). The Mahler conjecture is related to this problem and it asks for the minimum of the volume product

$$\mathcal{P}(K) = \mathrm{vol}_n(K)\mathrm{vol}_n(K^*).$$

One may notice $\mathcal{P}(K)$ is invariant under the action of nonsingular linear transformations. Thus it makes sense to consider the quantities $\max_K \mathcal{P}(K)$, $\min_K \mathcal{P}(K)$, where the maximum and minimum is taken over all symmetric, convex bodies in $\mathbb{R}^n$. In 1939 Santaló [Sa] proved that the maximum of $\mathcal{P}(K)$ is attained on the Euclidean ball. About the same time Mahler conjectured that the minimum should be attained on the unit cube Q, (the above inequality is sometimes called the reverse Santaló inequality).

Mahler [Ma2] himself proved the conjectured inequality in $\mathbb{R}^2$, *but the question is still open even in the three-dimensional case.* In the n-dimensional case, the conjecture has been verified for some special classes of bodies such as unconditional bodies [Me, Re2, SR], convex bodies having hyperplane symmetries which fix only one common point [BF], zonoids [KBH, GMR, Re1], bodies of revolution [MR3], and bodies with some positive curvature assumption [GM, RSW, S]. Bourgain and Milman, [BM], proved the isomorphic version of the conjecture. That is, there exists an absolute constant c such that $\mathcal{P}(K) \geq c^n \mathcal{P}(Q)$, for all convex bodies K. See [Mak, RZ, T1, T2, T3] for detailed discussions of the Mahler problem and properties of the dual volume.

3.1. **Duality on Lipshitz-free Banach spaces.** Following ideas from [GK], in [AFZ1] we consider a discrete version of the volume product. Consider a metric space $M = \{a_1, \ldots, a_{n+1}\}$, with metric d. We create a version of duality for metric spaces using Lipshitz functions on M. Note that the standard definition of duality is via the scalar product and thus based on the linear structure, which is not necessary available in a metric space and clearly not available in M. More precisely, consider the linear space $M^\sharp$ of Lipschitz functions f on M, with the restriction that $f(a_{n+1}) = 0$, equipped with a norm

$$\|f\|_{\mathrm{Lip}} = \max_{a_i \neq a_j} \frac{f(a_i) - f(a_j)}{d(a_i, a_j)}.$$

Note that each function f on M is just a set of n values $f(a_n)$, and thus we can identify $M^\sharp$ with $\mathbb{R}^n$ by assigning to a function $f \in M^\sharp$ a vector $f = (f(a_1), \ldots, f(a_n)) \in \mathbb{R}^n$. This allows us to embed the space into a polytope to better understand the geometry of $M^\sharp$ by visualizing the unit ball of $\|f\|_{\mathrm{Lip}}$ (denote it by B_{Lip}). In fact, we can derive an exact formula for the norm functional $\|f\|_{B_{\mathrm{Lip}}}$, corresponding to B_{Lip} on $\mathbb{R}^n$. Let us denote $d_{i,j} = d(a_i, a_j)$, then

$$\|f\|_{B_{Lip}} = \max_{i \neq j} \frac{f_i - f_j}{d_{i,j}} = \max_k f \cdot V_k, \text{ for } f \in \mathbb{R}^n,$$

Where V_k are the vectors $\pm \dfrac{e_i}{d_{i,n+1}}$ for $i = 1, \ldots n$, $j = n+1$, and $\dfrac{e_i - e_j}{d_{i,j}}$ for $i \neq j \in \{1, \ldots n\}$, where $e_1, \ldots, e_n$ is a standard basis of $\mathbb{R}^n$. Thus we have a very clear picture of B^*_{Lip}: it is just the convex hull of the points V_k described above. Notice that there are $2n + n(n-1) = n(n+1)$ vectors in V_k, and the vectors have a very specific form. Thus, we can define the volume product for the metric space as $\mathcal{P}(M) = \mathcal{P}(B_{\mathrm{Lip}})$ and examine the extreme values for the restricted class of polytopes defined in this way.

It is interesting to note that the maximum for the volume product in this space does not follow from the Santaló result on the maximality of the Euclidean ball. For a fixed number of points in a metric space M we always get a polytope with a bounded number of vertices, thus the Euclidean ball is never one of them. However, if we consider B^*_{Lip} for the metric space $M = \{0, 1, 2, \ldots, n+1\}$ where $d(i,j) = |i-j|$, then $\mathcal{P}(M) = \mathcal{P}(B_1^n) = \frac{4^n}{n!}$. So it is reasonable to conjecture that the minimal product volume possible coincides with Mahler's conjecture.

Let's consider another finite metric space M where the distance between any two points is constant. Then call $B^*_{\mathrm{Lip}} = K$ for this metric space. We conjecture the following:

Conjecture: *For fixed number of points in M, $\mathcal{P}(B_1^n) \leq \mathcal{P}(M) \leq \mathcal{P}(K)$.*

Utilizing the techniques of Symmetrizations and Shadow systems introduced by Rogers and Shephard [RS], further developed and applied to to the case of volume product by Campi and Gronchi [CG], with the more recent developments by Meyer and Reisner [MR1], and by Fradelizi, Meyer, and Zvavitch [FMZ], we were able to show the following in [AFZ1].

Theorem 4. *The maximum volume product for a metric space of three points is* 9.

Theorem 5. *The minimum volume product for a metric space of four points is $\mathcal{P}(B_1^3)$.*

3.2. **Future work.** Currently, the conjecture is open starting from the case of 6 elements and, clearly, the most interesting results correspond to the case of metric spaces with a large number of components (i.e. to large data structures). I am also working on a number of particular cases of the conjecture studying metric spaces containing just a few points or when the metric space has a special additional geometric structure, such as being a tree.

3.3. **Volume product on Polytopes.** In [AFZ2] we examine the volume product for classes of restricted polytopes extending the ideas above and work done in [MR1]. In particular, we found another proof of the following theorem first proven in [MR2].

Theorem 6. *The regular N-gon has maximal volume product among all origin symmetric polygons with N vertices.*

We then explore natural extensions from [MR1] where they studied bodies with few vertices compared to the dimension. Define $\mathbb{P}^n_k$ be the set of all polytopes of dimension n with at least k vertices. We use shadow systems and computations to find the following

Theorem 7. *The maximal volume product in $\mathbb{P}^3_5$ occurs for the the bi-pyramid with an equilateral triangular base.*

Theorem 8. *Let K be an origin symmetric body in $\mathbb{P}^3_8$. Then the maximal volume product of such bodies is the double cone with line segments for the apex in a regular orientation.*

3.4. **Future Work.** I am currently attempting to extend these results for dimension n. That is, we conjecture that maximal bodies in $\mathbb{P}^n_{n+2}$ are double cones with maximal base in $\mathbb{P}^{n-1}_n$. For symmetric bodies in $\mathbb{P}^n_{2n+4}$ we conjecture the maximum is a double cone whose base is maximal in $\mathbb{P}^{n-1}_{2n}$ and whose apex are line segments in a regular orientation. It may also be possible to increase the number of points in either $\mathbb{R}^3$ or in general.

References

[AHZ] M. Alexander, M. Henk, and A. Zvavitch, *A discrete version of Koldobsky's slicing inequality*, Isreal Journal of Mathematics, to appear, `https://arxiv.org/pdf/1511.02702v1`

[AFZ1] M. Alexander, M. Fradelizi, and A. Zvavitch, *Volume Product in Lipshitz-free Banach spaces*, in preparation

[AFZ2] M. Alexander, M. Fradelizi, and A. Zvavitch, *Polytopes of Maximal Volume Product*, in preparation

[BF] F. Barthe and M. Fradelizi, *The volume product of convex bodies with many hyperplane* symmetries, Amer. J. Math. 135 (2013), no. 2, 311-347.

[Bo1] J. Bourgain, *On high dimensional maximal functions associated to convex bodies*, Amer. J. Math. 108 (1986), 1467-1476.

[Bo2] J. Bourgain, *Geometry of Banach spaces and harmonic analysis*, Proceedings of the International Congress of Mathematicians (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, 871–878.

[BM] J. Bourgain, V. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbb{R}^n$, Invent. Math. 88 (1987), 319–340.

[Bu] H. Busemann, *A theorem of convex bodies of the Brunn-Minkowski type,* Proc. Nat. Acad. Sci. U.S.A. 35, (1949), 27-31.

[BP] H. Busemann, C.M. Petty, *Problems on convex bodies*, Math. Scand. 4 (1956), 88–94.

[CG] S. Campi, P. Gronchi, *On volume product inequalities for convex sets,* Proc. Amer. Math. Soc. 134 (2006), no. 8, 2393-2402.

[FMZ] M. Fradelizi, M. Meyer and A. Zvavitch *An application of shadow systems to Mahler's conjecture,* Discrete & Computational Geometry, 48 (2012), no. 3, 721–734.

[Ga] R.J. Gardner, it Geometric tomography. Second edition. Encyclopedia of Mathematics and its Applications, **58**. Cambridge University Press, Cambridge, 2006.

[GGZ] R.J. Gardner, P. Gronchi, and C. Zong, *Sums, projections, and sections of lattice sets, and the discrete covariogram.* Discrete & Computational Geometry, 34 (2005), no. 3, 391–409.

[GKS] R.J. Gardner, A. Koldobsky, Th. Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Annals of Math. 149 (1999), 691–703.

[GPV] A. Giannopoulos, G. Paouris, and B. Vritsiou, *The isotropic position and the reverse Santaló inequality*, BH. Isr. J. Math. (2014) 203: 1

[GK] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces.* Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday. Studia Math. 159 (2003), no. 1, 121–141.

[GM] Y. Gordon and M. Meyer, On the Minima of the Functional Mahler Product, Houston J. Math. 40 (2014), no. 2, 385–393.

[GMR] Y. Gordon, M. Meyer and S. Reisner, *Zonoids with minimal volume-product - a new proof,* Proc. Amer. Math. Soc., 104, (1988), 273–276.

[KBH] K. J. Böröczky and D. Hug, *Stability of the reverse Blaschke-Santaló inequality for zonoids and applications*, Adv. in Appl. Math. 44 (2010), no. 4, 309–328

[Kl] B. Klartag, *On convex perturbations with a bounded isotropic constant,* Geom. and Funct. Anal. (GAFA), Vol. 16, Issue 6 (2006) 1274–1290.

[Ko1] A. Koldobsky, Fourier Analysis in Convex Geometry, Math. Surveys and Monographs, 116, AMS, Providence RI 2005.

[Ko2] A. Koldobsky, *A* $\sqrt{n}$ *estimate for measures of hyperplane sections of convex bodies.* Adv. Math. 254, (2014), 33–40.

[Ko3] A. Koldobsky, *Slicing inequalities for subspaces of* L_p*,* arXiv:1310.8102.

[Ko4] A. Koldobsky, *Estimates for measures of sections of convex bodies,* arXiv:1309.6485.

[Ko5] A. Koldobsky, *A hyperplane inequality for measures of convex bodies in* $\mathbb{R}^n, n \leq 4$, Dicrete Comput. Geom. 47 (2012), 538–547.

[Ko6] A. Koldobsky, *A hyperplane inequality for measures of unconditional convex bodies,* arXiv:1312.7048.

[KoZ] A. Koldobsky, A. Zvavitch, *An isomorphic version of the Busemann-Petty problem for arbitrary measures,* Geometriae Dedicata, to appear.

[Ku] Greg Kuperberg, *From the Mahler conjecture to Gauss linking integrals*, Geom. Funct. Anal. 18 (2008), no. 3, 870–892.

[Ma1] K. Mahler, *Ein Minimalproblem für konvexe Polygone*, Mathematica (Zutphen) B7 (1939), 118–127.

[Ma2] K. Mahler, *Ein Ubertragungsprinzip für konvexe Korper. Casopis Pyest. Mat. Fys. 68, (1939). 93–102.*

[Mak] E. Makai, Jr., *The recent status of the volume product problem*, available from http://www.renyi.hu/ makai/jurata.pdf

[Me] M. Meyer, *Une caractÂ«erisation volumique de certains espaces normes de dimension finie*. Israel J. Math. 55 (1986), no. 3, 317–326.

[MR1] M. Meyer and S. Reisner, *Shadow systems and volumes of polar convex bodies,* Mathematika 53 (2006), no. 1, 129–148 (2007).

[MR2] M. Meyer and S. Reisner, *On the volume product of polygons*. Abh. Math. Semin. Univ. Hambg. 81 (2011), no. 1, 93–100. 52A20 (52A38)

[MR3] M. Meyer and S. Reisner, *Shadow systems and volumes of polar convex bodies*, Mathematika 53 (2006), no. 1, 129–148 (2007)

[MS] V.D. Milman, G. Schechtman, *Asymptotic theory of finite - dimensional normed spaces*, Springer Lecture Notes 1200, 1986

[N] Fedor Nazarov, *The Hörmander proof of the Bourgain-Milman theorem*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 2050, Springer, Heidelberg, 2012, pp. 335–343

[NPRZ] F. Nazarov, F. Petrov, D. Ryabogin, and A. Zvavitch, *A remark on the Mahler conjecture: local minimality of the unit cube*, Duke Mathematical Journal, 154 (2010), no 3, 419–430.

[R] O. Regev, *A Note on Koldobsky's Lattice Slicing Inequality*, https://arxiv.org/pdf/1608.04945v1.pdf

[Re1] S. Reisner, *Zonoids with minimal volume–product*, Math. Zeitschrift 192 (1986), 339–346.

[Re2] S. Reisner, *Minimal volume-product in Banach spaces with a 1-unconditional basis*, J. London Math. Soc. (2) 36 (1987), no. 1, 126–136

[RSW] S. Reisner, C. Schütt, and E. M. Werner, *Mahler's conjecture and curvature*, Int. Math. Res. Not. IMRN 1 (2012), 1–16.

[RS] C.A. Rogers and G.C. Shephard, *Some external problems for convex bodies*, Mathematika 5 (1958), 93–102.

[RYZ] D. Ryabogin, V. Yaskin, and N. Zhang *Unique determination of convex lattice sets*, Discrete and Comp. Geometry, to appear

[RZ] D. Ryabogin and A. Zvavitch, *Analytic methods in convex geometry,* in Analytical and Probabilistic Methods in the Geometry of Convex Bodies, IMPAN Lecture Notes, vol. 2, Warsaw (2014), 87-183.

[SR] J. Saint-Raymond, *Sur le volume des corps convexes symetriques,* Sem. d'Initiation a l'Analyse, 1980-1981, no.11.

[Sa] L.A. Santaló, *Un invariante afin para los cuerpos convexos del espacio de n dimensiones,* Portugal J. Math. 8 (1949), 155–161.

[S] A. Stancu, *Two volume product inequalities and their applications*, Canad. Math. Bull. 52 (2009), no. 3, 464–472

[T1] T. Tao, *Santaló's inequality*, http://terrytao.wordpress.com/2007/03/08/open-problem-the-mahler-conjecture-on-convex-bodies/.

[T2] T. Tao, *Structure and Randomness: Pages from Year One of a Mathematical Blog,* American Mathematical Society, Providence, RI, 2008.

[T3] T. Tao,, *Structure and randomness*, American Mathematical Society, Providence, RI, 2008. Pages from year one of a mathematical blog.

[Zh] Gaoyong Zhang, *A positive answer to the Busemann-Petty problem in four dimensions*, Annals of Math. **149** (1999), 535–543.

[Z1] A. Zvavitch, *Gaussian Measure of Sections of convex bodies*, Adv. Math., **188/1** (2004), 124–136.

[Z2] A. Zvavitch, *The Busemann-Petty problem for arbitrary measures*, Math. Ann. 331, (2005), 867–887.