1. Show that any entire function $f$ satisfying both $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z$ must be a constant function. Are both conditions necessary? Why or why not?

2.(a). Show that $g(z) = \sum_{k=1}^{\infty} \frac{1}{k^2 + z}$ is analytic in the half plane $\text{Re}(z) > 0$.

(b). Show in fact that $g$ is meromorphic in $\mathbb{C}$.

3. With $-\pi \leq \arg w \leq \pi$ and $|z| = 1$ show that
\[
\arg \left( \frac{z - 1}{z + 1} \right) = \begin{cases} 
\frac{\pi}{2} & \text{if } \text{Im}(z) > 0 \\
-\frac{\pi}{2} & \text{if } \text{Im}(z) < 0 
\end{cases}
\]

4. Let $\gamma$ be a fixed smooth simple closed contour. Use the function
\[
J(\lambda) = \frac{1}{2\pi i} \int_{\gamma} \frac{(f + \lambda g)'}{f + \lambda g} \, dz, \quad 0 \leq \lambda \leq 1
\]
to prove Rouché’s Theorem.

Hint: note that $J$, as a function of the real parameter $\lambda$, has several elementary properties.

5. Show that
\[
\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx = \pi \frac{e^{-a}}{a}, \text{ for } a > 0.
\]

6. Using appropriate limit theorems, compute
\[
\lim_{n \to \infty} \int_{a}^{\infty} \frac{ndx}{1 + n^2 x^2}
\]
for $a < 0, a = 0, a > 0$. 
7. (a) What does it mean to say ‘the function $f : [0, 1] \to \mathbb{R}$ is essentially bounded’?

(b) A function $F : [0, 1] \to \mathbb{R}$ is said to be a Lipschitz function whenever there is a constant $K > 0$ so that for any $x, y \in [0, 1],$

$$|F(x) - F(y)| \leq K|x - y|$$

(c) Using what you know about differentiability (in particular, the Lebesgue-Vitali theory), show that if $F : [0, 1] \to \mathbb{R}$ is a Lipschitz function, then the derivative $F'$ of $F$ exists almost everywhere, is essentially bounded and that for any $x \in [0, 1]$

$$F(x) = F(0) + \int_0^x F'(t)dt.$$ 

8. (a) Show that if $(M, d)$ is a metric space and if $(f_n)$ is a sequence of continuous real-valued functions defined on $M$ that converges uniformly to $f : M \to \mathbb{R}$, then $f$ is continuous.

(b) Give an example of a sequence $(f_n)$ of continuous real-valued functions defined on the closed unit interval $[0, 1]$ such that $(f_n(x))$ converges for each $x \in [0, 1]$ yet $\lim_{n} f_n(x) \equiv f(x)$ is not continuous. (A suitable set of pictures of the graphs of the $f_n$’s will do.)

9. Let $K$ be a bounded subset of $L^1[0, 1]$, the space of absolutely integrable, Lebesgue measurable functions defined on $[0, 1]$. We say that $K$ is uniformly integrable if given $\epsilon > 0$, there is a $\delta > 0$ so if $E$ is a measurable subset of $[0, 1]$ with $m(E) \leq \delta$ ($m$ is Lebesgue measure) then $\int_E |f|dm \leq \epsilon$ for all $f \in K$.

(i) Show that the closed unit ball $B$ of $L^2[0, 1]$ is uniformly integrable.

(ii) Give an example of a bounded subset of $L^1[0, 1]$ that is NOT uniformly integrable.
10. Let $f : [0, 1] \to \mathbb{R}$ be a bounded measurable function. Show that there exists a sequence $(s_n)$ of simple measurable functions, $s_n : [0, 1] \to \mathbb{R}$, such that $(s_n)$ converges uniformly to $f$ on $[0, 1]$.

(a) Can you also have $|s_n(x)| \leq |f(x)|$ for each $x$ in (a)?

(b) Can you have $|s_n(x)| \leq |f(x)|$ for each $x$ and $(|s_n(x)|)$ be increasing for each $x \in [0, 1]$?.