Complex Variables
Qualifying Examination, Jan. 10, 2005

1. Given three vertices of a parallelogram z_1, z_2, z_3 in $\mathbf{C}$, find the fourth vertex z_4, opposite to the vertex z_2, in terms of the other three vertices.

2. Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function. Calculate each of the following integrals, in terms of the function f and its derivatives. (Here, $C_s(b) = \{z \in \mathbf{C} \; : \; |z-b| = s\}$.)

(a). $$\frac{1}{2\pi i}\int_{C_r(a)} \frac{f^2(z)}{(z-a)^2}dz,$$

(b). $$\left[\ \frac{1}{2\pi i}\int_{C_r(a)} \frac{f(z)}{z-a}dz\ \right]^2,$$

(c). $$\frac{1}{2\pi i}\int_{C_r(a)} \frac{f(z)}{(z-a)^2}dz$$

(d). $$\frac{1}{2\pi i}\int_{C_r(a)} \frac{f^2(z)}{z-a}dz.$$

3. Let f be a meromorphic function on $\mathbf{C}$, having poles at the three points $z = 1+3i, 3-4i,$ and 5, as well as one removable singularity at $z = 3$. In each case below, either provide the requested quantity (with explanation) or explain why not enough information has been provided to find this quantity.

(a). $\limsup_{n\to\infty}(\frac{|f^{(n)}(3+i)|}{n!})^{1/n}$.
(b). $\lim_{z\to 5} |f(z)|$.
(c). $\lim_{z\to 1+3i}(z-1-3i)f(z)$.
(d). $\lim_{z\to\infty} |f(z)|$.

4. Let $f : D \to D$ be an analytic function, where D is the open unit disc in $\mathbf{C}$. Suppose that $f(1/4) = -2/3$. Is it possible for $f(1/3) = 2/3$? Explain your reasoning.

5. Let $f : D \to D$ be an analytic function, where D is the open unit disc in $\mathbf{C}$. Suppose that there is a positive number $\delta > 0$ such that for every θ, $|\theta| < \delta$, $\lim_{z\to e^{i\theta}} f(z) = 0$. Prove that $f \equiv 0$ on D.

6. (a). State the Cauchy-Riemann equations and prove that if f is differentiable at a point $b \in \mathbf{C}$, then the Cauchy-Riemann equations hold at b.

(b). Let $f : U \to \mathbf{C}$ be an analytic function on the domain U such that $\operatorname{Re} f(z) = (\operatorname{Im} f(z))^2$ for all $z \in U$. Prove that f is a constant function.

7. Find all real numbers b so that the following integral exists, and for these b evaluate this integral:

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + bx + 1} dx.$$

8. Let $P(z) = 2z^4 + 5z^2$ and $Q(z) = z^4 + 10z^2 + 1$. Prove that P and Q have the same number of zeros inside the open unit disc as well as the same number of zeros outside the unit disc but inside the disc of radius 4 centered at 0.

9. Find the linear fractional transformation which transforms the points $-1, 0, 1$ respectively into the points $1, i, -1$, and explain what the upper half-plane becomes in this mapping.

10. Find the domains of convergence of the given series.

(a). $$\sum_{n=0}^{\infty} \left(z^n + \frac{1}{2^n z^n} \right),$$

(b). $$\sum_{n=0}^{\infty} \frac{(-1)^n}{z+n}.$$