ALL

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Qualifying Examination
Complex Variables
Fall, 2000

Note: Throughout, $\mathcal{H}(U)$ denotes analytic functions on an open subset U of the complex plane $\mathbb{C}$. Also, throughout, D denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Good luck!

1. Let $\mathbb{C}^\infty$ denote the extended complex plane. Recall that a *linear fractional transformation* is a mapping $T : \mathbb{C}^\infty \to \mathbb{C}^\infty$ of the form

$$T(z) = \frac{az+b}{cz+d}.$$

a. Show that $T(0) \neq T(1)$ if and only if $ad - bc \neq 0$.

b. Suppose that T is a linear fractional transformation such that $T(0) = 1, T(1) = i$, and $T(\infty) = 0$. Find $T(i)$ and describe $T(\mathbb{R})$.

2. Let $f \in \mathcal{H}(U)$, where U is an open subset of $\mathbb{C}$ containing the closed unit disc $\overline{D}$.

(a). Suppose that $|f(z)| < 1$ for all z, $|z| = 1$. Prove that for all $k \in \mathbb{N}$, there are exactly k solutions (counting multiplicity) to the equation $f(z) = z^k$ in D.

(b). What conclusion can you draw if $|f(z)| \leq 1$ on $|z| = 1$? Justify your answer.

3. Let $U \subset \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a function which has a complex derivative $f'(z_0)$ at a point $z_0 \in U$. Derive the Cauchy-Riemann equations for f at the point z_0.

4. Find the Laurent series of $f(z) = z - \frac{1}{z} + exp(1/z)$ about 0.

5. Let $f : D \to D$ be an analytic function. Suppose that $f(\frac{1}{2}) = 0$. Find the largest value which $|f'(\frac{1}{2})|$ can have, and well as all functions f which have this value.

6. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be entire functions. Prove the following:

$$\frac{1}{2\pi i} \int_{|w|=1} \frac{f(w)g(z/w)}{w} dw = \sum_{n=0}^{\infty} a_n b_n z^n,$$

for all $z \in \mathbb{C}$, $|z| < 1$.

Use this to prove that $\frac{1}{2\pi i} \int_{|w|=1} \frac{e^{w+\frac{1}{w}}}{w} dw = \sum_{n=0}^{\infty} (\frac{1}{n!})^2$.

7. Let f have a pole of order 2 at the point $z = a$. Calculate the residue $\text{Res}(f;a)$ of f at a. Hence or otherwise, evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx.$$

8. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial. In each case, explain what the limit is:

$$a. \lim_{n\to\infty} \int_{\{|z|=n\}} \frac{P'(z)}{P(z)} dz.$$

$$b. \lim_{n\to\infty} \int_{\{|z-n|=1\}} \frac{P'(z)}{P(z)} dz.$$

9. Let $f \in \mathcal{H}(D)$ be such that the Taylor series of f about 0 has radius of convergence equal to 1. In each case, either prove or disprove the assertion.

a. There exists such an f which also satisfies $|f(z)| \to \infty$ as $|z| \to 1^-$.
b. There exists such an f such that f is also continuous on the closed unit disc $\overline{D}$.
c. For each such f, there exists a point z_0, $|z_0| = 1$, such that f cannot be analytically continued 'over' z_0.

10. Let $f : D \to \mathbb{C}$ be given by

$$f(z) = exp(\frac{z+1}{z-1}).$$

In each case, explain why f does, or why f does not, satisfy the condition:

a. $f \in \mathcal{H}(D)$.
b. $f \in \mathcal{H}^\infty(D)$, the space of *bounded* analytic functions on D.
c. $f \in \mathcal{A}(D)$, the space of analytic functions on D which are *continuous* on $\overline{D}$.