Qualifying Examination
Complex Variables
Fall, 2002

1. Give the Laurent series expansion **about 1** for the function

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

and describe the region on which your expansion converges.

2. a) Find a conformal equivalence of $\{z \in \mathbb{C} \ : \ \text{Re } z > 0\}$ with the open unit disc Δ such that $z = 1$ goes to $z = 0$.

b) Find a conformal equivalence of $\{z \in \mathbb{C} \ : \ \text{Re } z > 0\}$ with itself such that $z = 1$ goes to $z = 2$.

3. Provide a careful drawing of each of the following:
a) $\{z \ : \ |z-1| = |z-i|\}$.
b) $\{z \ : \ \text{Arg } z = \pi/3\}$.
c) $\{z \ : \ \text{Re } (z-1) = |z|\}$.

4. Prove that all the roots of the equation

$$z^6 - 5z^2 + 10 = 0$$

lie in the annulus $1 < |z| < 2$.

5. Prove that there is no function f which is analytic on Δ such that $\lim_{|z|\to 1} |f(z)| = \infty$.

6. Let f and g be entire functions which satisfy

$$f(0) = g(0) \neq 0 \quad \text{and} \quad |f(z)| \leq |g(z)| \, \forall z \in \mathbb{C}.$$

Prove that $f = g$.

7. Find $\int_0^{2\pi} \frac{d\theta}{3\sin\theta + 5}$.

8. Is there a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which is analytic in Δ but in no larger disc centered at the origin, such that $\sum_{n=0}^{\infty} |a_n| < \infty$? Is there an analytic function on Δ which is bounded on Δ but which is not continuous on $\overline{\Delta}$?

9. We say that an entire function f is of *exponential type* if there are $C > 0$ and $c > 0$ such that

$$|f(z)| \leq Ce^{c|z|} \qquad (*)$$

for every $z \in \mathbb{C}$. Prove that the set of entire functions of exponential type is an algebra. (Namely, show that if f and g are of exponential type, then so are $f + g$ and fg.) Show that if f is an entire function which satisfies $\sup_n |f^{(n)}(0)|^{1/n} < \infty$, then f is of exponential type. Prove that the converse implication also holds. Let P be a complex polynomial. Show that for every $c > 0$, there is some constant $C > 0$ such that (*) holds.

10. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(0) = 0$ and $f(z) = 1$ for $z \neq 0$.
a) Prove that there is a sequence of complex polynomials (P_n) such that for each $z \in \mathbb{C}$, $P_n(z) \to f(z)$ as $n \to \infty$.
b) Prove that no such sequence (P_n) can be such that $\sup_{n, z \in \Delta} |P_n(z)| < \infty$.