# Qualifying Examination
# Complex Variables
# Fall, 2003

1. Prove that if $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then the points $z_1, z_2, z_3$ are the vertices of an equilateral triangle inscribed in the unit circle.

2. Find all the values of the following powers:

(a) $1^{\sqrt{2}}$;

(b) $2^i$;

(c) $i^i$;

(d) $\left(\frac{1-i}{\sqrt{2}}\right)^{1+i}$;

(e) $(3-4i)^{1+i}$.

3. Are the functions

$$(a) \quad \frac{1}{1-z}, \qquad (b) \quad \frac{1}{1+z^2}$$

continuous inside the unit disk $|z| < 1$? Are they uniformly continuous?

4. (a) Prove or disprove: The function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^z$ is onto.

(b) Prove or disprove: The function $g : \mathbb{C} \to \mathbb{C}$ given by $g(z) = e^z + e^{-z}$ is onto.

5. Find a conformal mapping $f : D \to U$, where $D$ is the open unit disk $|z| < 1$ and $U = D \setminus \{z = x : x \leq 0\}$. (Recall that a mapping $f : D \to U$ is said to be *conformal* if $f$ is analytic, one-to-one, and onto.)

6. Does there exist a function analytic at the point $z = 0$ and assuming at the points $z = \frac{1}{n}$, $n = 1, 2, \ldots$, the values:

(a) $0, 1, 0, 1, \ldots, 0, 1, \ldots$;

(b) $0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, \ldots, 0, \frac{1}{2k}, \ldots$;

(c) $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \ldots, \frac{1}{2k}, \frac{1}{2k}, \ldots$;

(d) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$.

7. Find the singular points of the functions, explain their nature and investigate the behavior of the function at infinity.

(a)

$$\frac{z^4}{1+z^4},$$

(b)

$$\frac{e^z}{z(z^2+4)^2}.$$

8. Find the sets on which the given sequences converge uniformly:

(a) $\left\{\frac{1}{1+z^n}\right\}$;

(b) $\left\{\frac{z^n}{1+z^{2n}}\right\}$;

(c) $\left\{\frac{\sin nz}{n}\right\}$.

9. Evaluate the integral

$$\int\limits_C \frac{z\,dz}{(z-1)(z-2)^2},$$

where $C$ is the circle $|z-2|=\frac{1}{2}$.

10. Using Rouché's theorem find the number of roots of the equation

$$z^9 - 2z^6 + z^2 - 8z - 2 = 0$$

lying within the circle $|z| < 1$.