Qualifying Examination
Complex Variables
August, 2004

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Define g by

$$g(\lambda) = \int_{|z-i|=2} \frac{f(z)}{(z-\lambda)^2} dz.$$

(a). Find the largest open subset of $\mathbb{C}$ in which g is analytic.
(b). Calculate $g(2+i/2)$ and $g'(0)$.

2. Use Rouché's to show that $z^6 + 4z^2 - 1$ has exactly two zeros in the unit disk.

3. Let $P(z) = c(z-a)^m(z-b)^n$, where c is a constant. Suppose that

$$\frac{1}{2\pi i}\int_{|z|=2} \frac{P'(z)}{P(z)} dz = 3, \frac{1}{2\pi i}\int_{|z-4i|=3} \frac{P'(z)}{P(z)} dz = 1,$$

and

$$\frac{1}{2\pi i}\int_{|z-(2-i)|=1} \frac{P'(z)}{P(z)} dz = 0.$$

Give as much information as you can about a, b, c, m, and n.

4. Let $f : D(0;1) \to \mathbb{C}$ be a non-constant analytic function. Suppose that for all $z \in D(0;1), Re(f(z)) \geq 0$.
(a). Prove that, in fact, $Re(f(z)) > 0$ (i.e. show that the inequality is strict).
(b). Find a fractional linear transformation T that takes the set $\{z \ : \ Rez > 0\}$ onto $D(0;1)$ such that $T(1) = 0$.
(c). Suppose also that $f(0) = 1$. Prove that the following estimate holds for all $z \in D(0;1)$:

$$|f(z)| \leq \frac{1+|z|}{1-|z|}.$$

5. Let $K \subset U$ be a compact subset of the open set U. Define $\hat{K}_U := \{z \in U \ : \ \text{for every analytic function } f \text{ on } U, |f(z)| \leq \max_{w \in K} |f(w)|\}$.

(a). Prove that $K \subset \hat{K}_U$.

(b). Let $K = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Calculate $\hat{K}_U$ and $\hat{K}_{\mathbb{C}}$ where $U = \{z \in \mathbb{C} \ : \ 0 < |z| < 2\}$.

(c). Prove that $dist(K, \partial U) = dist(\hat{K}_U, \partial U)$.

6. Suppose that $f = u + iv$ is analytic on a domain $U \subset \mathbb{C}$, and that there is a *differentiable* function $h : \mathbb{R} \to \mathbb{R}$ such that $v(x, y) = h \circ u(x, y)$ for every $z = x + iy \in U$.

(a). Prove that f is a constant function.

(b). Suppose that h is merely assumed to be a function (i.e. assume that the differentiability assumption on h is removed). Does the conclusion in (a) still hold?

7. Define the function $\cos : \mathbb{C} \to \mathbb{C}$. Prove or disprove each of the following assertions about $\cos z$.

(a). For every $z \in \mathbb{C}$, $|\cos z| \leq 1$.

(b). The function cos is *onto* $\mathbb{C}$.

8. Find all possible entire functions $f : \mathbb{C} \to \mathbb{C}$ which are one-to-one, with $f(0) = 0$.

9. Find the Laurent series for the function

$$f(z) = \frac{1}{z(z-2)(z-4)}$$

that is valid in the annulus $\{z \ : \ 2 < |z| < 4\}$.

10. Prove that the coefficients c_n of the expansion

$$\frac{1}{1 - z - z^2} = \sum_{n=0}^{\infty} c_n z^n$$

satisfy the relation $c_n = c_{n-1} + c_{n-2}\,(n \geq 2)$. Find the radius of convergence of the series.