Complex Qualifying Exam August 1999

Answer all questions if possible.

1) Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function which is one to one.

a) Prove that $|f(z)| \to \infty$ as $|z| \to \infty$.

b) Prove that f is a polynomial.

2) For each of the functions f below, give all possible Laurent series expansions of f about the point $z = 0$ and for each Laurent series you find give the largest set on which your series converges to f.

a) Let $f(z)$ be the branch of $\text{Log}(1+z)$ on the domain $D = \mathbf{C} \sim [-1, -\infty)$ which has the value zero at $z = 0$.

b) Let $f(z)$ be the branch of $\text{Log}(1+z)$ on the domain $D = \mathbf{C} \sim [-1, -\infty)$ which has the value $2\pi i$ at $z = 0$.

3) Evaluate the following:

(a)

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-2i\theta}}{(e^{i\theta} - 2)^2} d\theta$$

(b) **For each natural number** n

$$\frac{1}{2\pi i} \int_{|z|=1} (z + \frac{1}{z})^n dz$$

4) State the Argument Theorem (which gives a formula for certain integrals of the form $\int_\gamma \frac{f'(z)}{f(z)} dz$) and **sketch** the proof.

5) Find all entire functions f such that f has a zero of order 2 at the origin, f(i) = -2, and $|f'(z)| \leq 6|z|$ or explain why such an f cannot exist.

6) Let

$$f(z) = \sum_{n=1}^{\infty} 1 - \cos(\frac{z}{n}).$$

Give a detailed proof that f is entire.

7) For each of the domains D below, assume f is an analytic function from D to $\mathbb{C}$ such that $\mathrm{Re}(f(z)) < 0$ for all $z \in D$ and $f(0) = -1$. Show that $|f'(0)| < 2$.

a) $D = \Delta(0, 1)$, the unit disk.
b) $D = \mathbb{C}$.

8) Let D be a domain in $\mathbb{C}$ (i.e., an open connected subset of $\mathbb{C}$) and let $f : D \to \mathbb{C}$ be a conformal map defined on D. Denote by $f(D)$ the image of D under the map f. Answer each of the following questions and justify your answer either with a counter example or as a consequence of appropriate theorems (which you should state clearly but need not prove).

(a) Is $f(D)$ always a domain?
(b) Define what it means for E to be a discrete subset of D. If E is discrete subset of D is $f(E)$ always a discrete subset of $f(D)$?
(c) If D is a simply connected domain is $f(D)$ always a simply connected domain?

9) Let $n \in \mathbb{N}$ and $a > 0$ be fixed. Consider the equation

$$e^z - az^n = 0.$$

a) Show that this equation has no solutions on the unit circle if a is not inside the interval $[e^{-1}, e]$.
b) How many solutions does the equation have inside the open unit disk, when $a > e$? (Prove your answer.)
c) How many solutions does the equation have inside the open unit disk, when $a < e^{-1}$? (Prove your answer.)

10) Let D be a domain in $\mathbb{C}$.
a) Let $u : D \to \mathbb{R}$ be a harmonic function. Show that if u^2 is a harmonic function on D then u is constant.
b) Let $f : D \to \mathbb{C}$ be an analytic function on D. Show that if $|f|^2$ is harmonic then f is constant.