Kent State University

Department of Mathematics and Computer Science

Qualifying Examination

COMPLEX VARIABLES

SPRING 1994

1. Evaluate the following integrals.

   a) $\oint_{|z|=2} z^2 e^{1/z} dz.$

   b) $\oint_{|z|=2} \frac{\cos z}{z^2(z-1)} dz.$

2. Find all possible Laurent expansions of $f(z) = \frac{1}{z^2 - 1}$ about $z = 1$.

3. Let f and g be entire functions which satisfy

   $$\text{(i)} \quad f(0) = g(0) \neq 0 \text{ and} \quad \text{(ii)} \quad |f(z)| \leq |g(z)|,$$

   for every complex number z. Prove that $f = g$.

4. Let C_1 be the circle with center 0 and radius 1, and let C_2 be the circle with center $\frac{1}{2}$ and radius $\frac{1}{2}$.

   Consider the function $f(z) = (z-1)^{-1}$.

   a) Determine the image under f of the region between C_1 and C_2.

   b) Determine the image under f of the region in the first quadrant between C_1 and C_2.

5. Determine all entire functions $f(z)$ which have the property that there exists a real number M such that $Re\{f(z)\} - x \leq M$ for all $z = x + iy \in \mathbb{C}$.

6. Let $|a_m| < 1, \; m = 1, 2, \ldots, n$ and

$$F(z) = \prod_{m=1}^{n} \left[\frac{z - a_m}{1 - \bar{a}_m z} \right].$$

Prove that if $|b| < 1$, the equation $F(z) = b$ has exactly n roots in the open unit disk.

7. Let f be analytic on an open set containing $\bar{\Delta} := \{z \in \mathbb{C} : |z| \leq 1\}$.

a) Show that if $|f(z)| < 1$ for every $|z| = 1$ then there is a unique $z_0 \in \Delta$ with $f(z_0) = z_0$. (Note: $\Delta = \{z \in \mathbb{C} : |z| < 1\}$).

b) Give an example to show that this conclusion no longer holds if we relax the condition on f to "$|f(z)| \leq 1$ for every $|z| = 1$".

c) Show that if $|f(z)| > 1$ for every $|z| = 1$ and $f(0) = 1$ then f must have a zero in Δ.

8. Let C be the unit circle, oriented counter clockwise. For any z in the complex plane, $|z| \neq 1$, evaluate

$$\int_C \frac{\bar{\zeta} d\zeta}{\zeta - z}.$$

9. Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and suppose that $f : \Delta \to \mathbb{C}$ is an analytic function which is also one-to-one. Assume that $f(0) = 0$, $f'(0) = 1$, and f is *not* the identity map on Δ. Prove that

a) $f(\Delta) \not\subseteq \Delta$,

b) $\Delta \not\subseteq f(\Delta)$.

10. Let $g(z) = \cos\left(\sqrt{z}\right) := \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(2k)!}$.

a) Prove that $g(z)$ is an entire function of z.

b) Find an infinite product representation of $g(z)$.