Numerical Analysis Qualifier

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January, 2004

INSTRUCTIONS: Do any (and only) 8 of the following 10 problems.

1. (Rounding-Error Analysis) Consider the evaluation in floating-point arithmetic of the finite sum

\[ \sum_{k=0}^{n} x_k = x_0 + x_1 + \cdots + x_n \]

from left to right. One can establish that

\[ \text{fl} \left( \sum_{k=0}^{n} x_k \right) = \sum_{k=0}^{n} x_k^* \]

where

\[ x_k^* = x_k (1 + \rho_k)^{n-k+1}, \quad k = 0, \ldots, n, \]

for some \( |\rho_0|, \ldots, |\rho_n| \leq \text{eps}. \)

(a) Using this and the inequality

\[ |(1 + \rho_1)^{\pm 1} \cdots (1 + \rho_m)^{\pm 1} - 1| \leq \frac{m \text{ eps}}{1 - m \text{ eps}} \]

(\( |\rho_1|, \ldots, |\rho_m| \leq \text{eps}, m \text{ eps} < 1 \)), derive a bound on the maximum relative error in \( \text{fl}(\sum_{k=0}^{n} x_k) \).

(b) What conclusions can you draw from this for series with positive terms versus alternating series?

(c) What conclusions can you draw from this for absolutely non-increasing \( (|x_0| \geq |x_1| \geq \cdots \geq |x_n|) \) series?

2. (Polynomial Interpolation) The function \( \cos x \) on \([-\pi/2, \pi/2]\) looks roughly like a downward opening parabola.

(a) Consider interpolating the function by a quadratic polynomial at \( x = -\pi/2, 0, \) and \( \pi/2 \). Determine a bound for the maximum error in this approximation.
(b) Consider interpolating both the function and its first derivative values at \( x = -\pi/2, 0, \) and \( \pi/2 \) by a polynomial of as low degree as possible. Determine a bound for the maximum error in this approximation.

(c) Give expressions for the polynomials in problems (a) and (b).

3. (Discrete Fourier Analysis) Given the discrete inner product

\[
[f, g] := \frac{1}{2N + 1} \sum_{k=0}^{2N} f(x_k)g(x_k), \quad x_k = \frac{2k\pi}{2N + 1}, \quad k = 0, \ldots, 2N,
\]

and the orthogonality relation

\[
[e^{inx}, e^{inx}] = \delta_{mn}, \quad m, n = -N, \ldots, N,
\]

(a) Derive the expression for the discrete Fourier coefficients \( \hat{f}(n) \) in the approximation

\[
f(x) \approx p_N(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}
\]

that minimizes the error with respect to the discrete norm induced by \([,\cdot]\).

(b) Discuss the Fast Fourier Transform for a vector of length \( N = 2^n \). What is it? What is it used for? Roughly how many operations does it require versus "brute force?"

4. (Least Squares) Given \( a, b \in \mathbb{R}^n \), solve

\[
\min_{z \in \mathbb{R}} \|az - b\|_2.
\]

5. (Singular Value Decomposition)

(a) Consider the column vector \( a \) as an \( n \times 1 \) matrix. Write out its singular value decomposition, showing explicitly the matrices that make it up.

(b) Consider the row vector \( a^T \) as a \( 1 \times n \) matrix. Write out its singular value decomposition, showing explicitly the matrices that make it up.

6. (The Arnoldi Process) Application of \( \ell \) steps of the Arnoldi process to the matrix \( A \in \mathbb{R}^{n \times n} \) with initial vector \( v \) yields the decomposition

\[
AV_\ell = V_{\ell+1}H_\ell.
\]
(a) Describe the matrices $V_\ell$, $V_{\ell+1}$ and $\tilde{H}_\ell$. How large are they? What are their properties? What is range($V_\ell$)?

(b) Present an algorithm for computing the decomposition (1).

7. (The GMRES method) The GMRES method is a popular iterative method based on the Arnoldi decomposition (1).

(a) Derive the GMRES method.

(b) Describe the computations required.

(c) What is the restarted GMRES method? Why is it used?

8. (Zeros of Orthogonal Polynomials) Let $p_0, p_1, p_2, \ldots$, be a family of monic orthogonal polynomials with respect to the inner product

$$ (f, g) = \int_{-1}^{1} f(x)g(x)w(x) \, dx, $$

where $w(x)$ is a nonnegative smooth function. Thus, $p_j$ is a polynomial of degree $j$ with leading coefficient one. Show that for each $j$, $p_j$ has only simple zeros, and they all are in the interval $[-1, 1]$.

9. (Recurrence Relation for Orthogonal Polynomials) Let $p_0, p_1, p_2, \ldots$, be the family of orthogonal polynomials of Problem 8.

(a) Give the form of the recurrence relation satisfied by the orthogonal polynomials.

(b) Show the existence of the recurrence relation. The proper values of the recurrence coefficients has to be provided.

10. (Quadrature) Consider the approximation of the integral $\int_{-1}^{1} f(x)w(x) \, dx$, where $w(x)$ is the weight function of Problem 8, by a sum of the form

$$ S_n f = \sum_{j=1}^{n} f(x_j)w_j. $$

Let the nodes $x_j$ be the zeros of the orthogonal polynomial $p_n$ of Problem 8.

(a) Give a linear system of equations for the weights $w_1, w_2, \ldots, w_n$, such that the quadrature rule is exact for all polynomials of degree at least $n - 1$. The system does not have to be solved.

(b) Show that the quadrature rule with the so determined weights is exact for all polynomials of degree at least $2n - 1$. 