Numerical Analysis Qualifier

prepared by
Chuck Gartland & Arden Ruttan
Kent State University
Fall, 1989

INSTRUCTIONS: Do any 10 of the following 11 problems.

1. (Conditioning)
   (a) Define the *mathematical condition number*, $\mathrm{cond}(f;x)$, of $f : D \subseteq X \to Y$ (X and Y normed linear spaces and D open) at $x \in D$.
   (b) Give a formula for $\mathrm{cond}(f;x)$ that is valid when f is differentiable at x.
   (c) Compute the condition number for the real function $f(x) = \sqrt{x+1} - \sqrt{x}$, $(0 \leq x)$. For what values of x would you say that this function is well-conditioned, ill-conditioned?

2. (Polynomial Interpolation)
   (a) Prove that the classical polynomial interpolation problem

   "given (x_i, f_i), $i = 0, \ldots, n$, $-1 \leq x_0 < \cdots < x_n \leq 1$, find $P \in \Pi_n$ such that $P(x_i) = f_i$, $i = 0, \ldots, n$"

   has a unique solution.
   (b) Let $L_n(f, x)$ denote this interpolant. Prove that if $f \in C^{n+1}[-1, 1]$, then for each $x \in [-1, 1]$ there is a number $\xi(x)$ in $(-1, 1)$ such that

   $$f(x) = L_n(f, x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

   (c) On the basis of the error formula above, what are "good values" for the abscissas $x_0, \ldots, x_n$? In what sense? How are they distributed in $[-1, 1]$?

3. (Divided Differences)
   (a) Give the definition/characterization of the n-th order divided difference $f[x_0, \ldots, x_n]$ and the recursion that it satisfies.
   (b) Derive the closed-form expression

   $$f[x_0, \ldots, x_n] = \sum_{k=0}^{n} \frac{f(x_k)}{\prod_{j \neq k}^{n} (x_k - x_j)}.$$

4. (Discrete Fourier Analysis) Given the discrete inner product

$$[f, g] := \frac{1}{2N+1} \sum_{k=0}^{2N} f(x_k)\overline{g(x_k)}, \qquad x_k = \frac{k\pi}{2N+1}, \quad k = 0, \ldots, 2N,$$

and the orthogonality relation

$$[e^{imx}, e^{inx}] = \delta_{mn}, \quad m, n = -N, \ldots, N,$$

(a) Derive the expression for the *discrete Fourier coefficients* $\hat{f}(n)$ in the approximation

$$f(x) \approx p_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

that minimizes the error with respect to the discrete norm induced by $[\cdot, \cdot]$.

(b) Discuss the *Fast Fourier Transform* for a vector of length $N = 2^n$. What is it? What is it used for? Roughly how many operations does it require versus "brute force?"

5. (Peano Kernels) Peano kernel functions can be used to represent linear functionals other than just error functionals for quadrature rules—which was our main application in the course. Derive the Peano kernel function for the "divided-difference functional"

$$f \mapsto f[x_0, x_1, x_2] \qquad (x_0 < x_1 < x_2)$$

and sketch it.

6. (Extrapolation)

(a) What does it mean for the family $\{\phi_k(h)\}_{k=0}^{\infty}$ to be an *asymptotic family* (as $h \to 0$)? Give an example.

(b) What does it mean for the formal expansion

$$\phi(h) \sim c_0 \phi_0(h) + c_1 \phi_1(h) + \cdots$$

to be an *asymptotic expansion*?

(c) Give the form of the asymptotic expansion for the composite Trapezoid rule. How is this proved/derived? For what can this be used (casually)?

7. (Numerical Linear Algebra)

(a) Let $\|\cdot\|$ be a norm on $\mathbf{C}^n$. Define the *matrix norm* $\|\cdot\|^*$ induced by $\|\cdot\|$.

(b) Let $\|\cdot\|_1^*$ be the matrix norm induced by the l_1 norm on $\mathbf{C}^n$. Prove that for any matrix $A \in \mathbf{C}^{n,n}$, $A = (a_{ij})_{i,j=1}^n$, one has

$$\|A\|_1^* = \max_{1 \le j \le n} \sum_{k=1}^{n} |a_{kj}|.$$

(c) Give computational formulas for the following vector and matrix norms: $\|x\|_2$, $\|x\|_\infty$, $\|A\|_F$, and $\|A\|_\infty$.

(d) Give "ball-park" operation counts for the following standard algorithms: general LU factorization, Cholesky decomposition, back-substitution, and orthogonal triangularization.

8. (Least Squares Problems)

   (a) Discuss how orthogonalization can be used to solve a full-rank, linear, least-squares problem (LS).

   (b) Discuss the conditioning of LS.

   (c) Discuss how the *normal equations* can be used to solve LS, and compare their conditioning with the orthogonalization approach.

9. (Nonlinear Systems and Unconstrained Optimization)

   (a) Prove that fixed-point iteration with a contraction mapping must converge (at least) Q-linearly, if at all.

   (b) We know that the Newton direction

   $$d = Df(x)^{-1}f(x)$$

   is a *descent direction* for the functional $h(x) := \|f(x)\|_2^2$. On the other hand, any unconstrained optimization problem $\min_{x\in\mathbf{R}^n} h(x)$ can be attacked via Newton's method applied to the gradient equation $\nabla h(x) = 0$, i.e.,

   $$x^{(k+1)} = x^{(k)} - \nabla^2 h(x^{(k)})^{-1}\nabla h(x^{(k)}).$$

   Show that in this case, positive definiteness of the *Hessian matrix* $\nabla^2 h(x^{(k)})$ is sufficient to guarantee that the Newton direction here is a descent direction for the original functional $h(x)$.

10. (Eigenvalue Problems)

    (a) Let A be a Hermitian $n \times n$ matrix, and assume that all of the eigenvalues of A are simple. Show that for *any* starting vector $x_0 \in \mathbf{C}^n$, the *power method* (or *simple vector iteration*) converges.

    (b) Discuss the conditioning of the eigenvalues for *normal*, *non-defective*, and *defective* matrices.

    (c) Given good approximations to some eigenvalues of a matrix A, how would you compute approximate eigenvectors associated with them?

11. (Ordinary Differential Equations)

    (a) In *nonlinear shooting* applied to the problem

    $$y'' = f(x, y, y'), \qquad y(a) = \alpha, \quad y(b) = \beta,$$

one frequently solves the ODE

$$y''(x;t) = f(x, y, y'), \qquad y(a) = \alpha, \quad y'(a) = t$$

for $y(b;t)$ and then uses Newton's method to solve $y(b;t) = \beta$. This method requires one to calculate $\partial y(b;t)/\partial t$. Derive an ODE that can be solved to obtain $\partial y(b;t)/\partial t$.

(b) In the case where the original problem is *linear*, that is, of the form

$$y'' = p(x)y' + q(x)y + r(x), \qquad y(a) = \alpha, \quad y(b) = \beta,$$

the situation is much simpler because the solution can be expressed as a combination of the solutions of two initial-value problems. Derive/verify this expression.