

## Numerical Analysis Qualifier

prepared by

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INSTRUCTIONS: Do any 8 of the following 9 problems.

1. (Rounding Errors) Consider the straightforward evaluation of the simple formula

$$y = f(x) = \sqrt{x-1}$$

in floating-point arithmetic.

- (a) Using leading-order error-propagation analysis and assuming that  $\sqrt{\cdot}$  is delivered to machine precision, derive an approximate expression for the relative error,  $(\tilde{y} - y)/y$ , in the computed result.
- (b) For what values of  $x$  do you expect poor relative accuracy? Is the problem the *mathematical* conditioning of  $f(x)$  or the *numerical* conditioning/stability of the evaluation algorithm? What can you say about the case where  $x$  is a machine number?

2. (Divided Differences)

- (a) Give the definition of *divided differences*.
- (b) Give the general expression for the divided-difference/Newton form of the polynomial of degree at most  $n$  that interpolates to  $f$  at the points  $x_0, \dots, x_n$ .
- (c) Use this form to compute the cubic polynomial that solves the *Hermite* interpolation problem

$$P_3(0) = 1, \quad P_3'(0) = -2, \quad P_3(1) = -2, \quad P_3'(1) = -8.$$

3. (Piecewise Polynomial Interpolation) A function  $L_\Delta$  is in the class  $\mathcal{L}_\Delta$  of “real piecewise-linear polynomials” (relative to a given partition  $\Delta : a = x_0 < x_1 < \dots < x_n = b$ ) if it satisfies

- (a)  $L_\Delta \in C[a, b]$ ,
- (b)  $L_\Delta|_{[x_i, x_{i+1}]} \in \Pi_1$ ,  $i = 0, \dots, n-1$ ,

that is,  $L_\Delta$  is a continuous function that coincides with a polynomial of degree at most *one* in each subinterval.

- (a) Show that  $\mathcal{L}_\Delta$  is a vector space, and determine its dimension.
- (b) The classical interpolation problem for  $\mathcal{L}_\Delta$  is “given  $(x_i, f_i)$ ,  $i = 0, \dots, n$ , find  $L_\Delta \in \mathcal{L}_\Delta$  such that

$$L_\Delta(x_i) = f_i, \quad i = 0, \dots, n.”$$

Prove that this problem has a unique solution.

- (c) Supposing that the data come from a twice continuously differentiable function, i.e.,  $f_i = f(x_i)$ ,  $i = 0, \dots, n$ , ( $f \in C^2[a, b]$ ), derive the error bound

$$\|f - L_\Delta(f)\|_\infty \leq \frac{1}{8} \|f''\|_\infty \|\Delta\|^2, \quad \|\Delta\| := \max_{j=0, \dots, n-1} |x_{j+1} - x_j|.$$

4. (Euler-MacLaurin Summation Formula, Extrapolation)

- (a) What is the Euler-MacLaurin summation formula? State it as precisely as you can.
- (b) In studying numerical quadrature, what is (are) the main application(s) of this result?
- (c) Assume that some quantity of interest (like a quadrature rule) is known to admit an asymptotic expansion of the form

$$T(h) \sim \tau_0 + \tau_1 h + \tau_2 h^2 + O(h^3), \quad \text{as } h \rightarrow 0,$$

where  $\tau_0$ ,  $\tau_1$ , and  $\tau_2$  are unknown real numbers. Construct from this an extrapolation formula for  $\tau_0$  based upon  $T(h)$  and  $T(h/2)$ .

- (d) Determine the asymptotic expansion for this formula.

5. (Cholesky Decomposition) Show that a symmetric positive definite matrix  $A$  has a Cholesky decomposition  $A = LL^T$  where  $L$  is lower triangular. Hint: use induction and the fact that if

$$A = \begin{bmatrix} A_1 & a \\ a^T & \alpha \end{bmatrix} \text{ and } b = A_1^{-1}a,$$

then

$$0 < [b^T, -1] \begin{bmatrix} A_1 & a \\ a^T & \alpha \end{bmatrix} \begin{bmatrix} b \\ -1 \end{bmatrix}.$$

6. (Matrix Norms and Condition Numbers)

- (a) Give computational formulas for the vector norms  $\|x\|_1$ ,  $\|x\|_2$ , and  $\|x\|_\infty$ .
- (b) For the matrix  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , compute the following norms:  $\|A\|_1$ ,  $\|A\|_2$ ,  $\|A\|_\infty$ , and  $\|A\|_F$ .
- (c) Define the *matrix condition number*,  $\kappa(A)$ , and prove that  $\kappa(A) \geq 1$  for any operator norm.
- (d) Consider the perturbed linear system

$$(A + \Delta A)(x + \Delta x) = b + \Delta b,$$

where  $A$  is assumed to be nonsingular, and  $\Delta A$  is “sufficiently small.” State (DO NOT DERIVE) the standard bounds on the relative error  $\|\Delta x\|/\|x\|$  in terms of (i) the relative perturbations  $\|\Delta A\|/\|A\|$  and  $\|\Delta b\|/\|b\|$ , and (ii) the relative residual  $\|r\|/\|b\|$  ( $r := b - A(x + \Delta x)$ ).

7. (Least Squares, Singular Value Decomposition) Consider the problem of finding a polynomial  $p(x)$  of degree  $n$  which minimizes  $\sum_{j=1}^m (p(x_j) - f_j)^2$ , where  $m > n$ ,  $x_1 < x_2 < \dots < x_m$ , and  $\{f_j\}_{j=1}^m$  is a given set of real numbers.

- (a) Show that this problem can be expressed as a *linear least squares problem* of the form: Find  $z \in \mathbb{R}^{n+1}$  which minimizes  $\|Az - b\|_2$ .
- (b) Let  $A = U\Sigma V^H$  be the *Singular Value Decomposition* of  $A$ , where  $U$  and  $V$  are *unitary* (of dimensions  $m \times m$  and  $(n+1) \times (n+1)$  respectively) and  $\Sigma$  is  $m \times (n+1)$  and of the form

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & O \end{bmatrix},$$

with  $\sigma_1, \dots, \sigma_r > 0$ . Determine the value of  $\min_{z \in \mathbb{R}^{n+1}} \|Az - b\|_2$  in terms of the singular values of  $A$  and the components of  $U^H b$ .

8. (Algebraic Eigenvalue Problems)

- (a) Describe the algorithms for the basic *power method* (simple vector iteration) and the *shifted* power method to compute an approximate eigenvector associated with the dominant eigenvalue of a given matrix  $A$ .
- (b) Suppose you know the following information about the eigenvalues of a given  $n \times n$  matrix  $A$ :

$$\lambda_1 = 2, \quad \lambda_2, \dots, \lambda_n \in [0, 1].$$

What can you say about the rate of convergence of the iterates of the basic algorithm to a normalized eigenvector  $x_1$  ( $\|x_1\|_\infty = 1$ ) associated with  $\lambda_1$ ? Roughly how many iterations do you expect to need to get three accurate decimal digits in the components of  $x_1$ ?

- (c) Consider now the same problem (same matrix) but using the shifted power method. What is the “optimal” shift to use, given the information you have about the spectrum of the matrix? What will the rate of convergence be in this case? How does the total work required compare with the unshifted, basic power method?

9. (Multistep Methods for ODEs) Let

$$\eta_{j+r} + a_{r-1}\eta_{j+r-1} + \dots + a_0\eta_j = h \{b_r f(x_{j+r}, \eta_{j+r}) + b_{r-1}f(x_{j+r-1}, \eta_{j+r-1}) + \dots + b_0 f(x_j, \eta_j)\}$$

be a linear multistep method for the ordinary differential equation

$$y' = f(x, y), \quad y(x_0) = y_0.$$

- (a) Define the *local discretization error*  $\tau(x, y; h)$  for this multistep method.
- (b) Define a *consistent* linear multistep methods of *order*  $p$ .
- (c) Consider a two-step method of the form

$$\eta_{j+2} + a_1\eta_{j+1} + a_0\eta_j = h \{b_1 f(x_{j+1}, \eta_{j+1}) + b_0 f(x_j, \eta_j)\}.$$

Determine values for  $a_1$ ,  $a_0$ ,  $b_1$ , and  $b_0$  so that this multi-step method is consistent of order 3.