Numerical Analysis Qualifier

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INSTRUCTIONS: Do any 8 of the following 12 problems.

1. (Polynomial Interpolation)

   (a) Given $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ in a real interval $[a,b]$ and associated real numbers $f_0, f_1, \ldots, f_n$, show that there is a polynomial p_n of degree at most n, such that

   $$p_n(x_i) = f_i, \qquad i = 0, 1, \ldots, n. \tag{1}$$

   (b) Show that the polynomial p_n of part (a) is unique.

   (c) Suppose that the numbers f_i of part (a) are defined by $f_i = f(x_i)$, $i = 0, 1, \ldots, n$, where f is a real-valued function that is differentiable arbitrarily many times on $[a,b]$. Moreover, let there be a constant M, such that

   $$\max_{a \le x \le b} \left| \frac{d^j}{dx^j} f(x) \right| \le M$$

   for all $j \ge 0$. Can it be shown without additional assumptions about the location of the x_i that $p_n(x)$ converges uniformly to $f(x)$ on $[a,b]$ as $n \to \infty$? Motivate.

2. (Divided Differences)

   (a) Define divided differences.

   (b) Express the polynomial in Problem 1(a) by Newton's interpolation formula. How are the coefficients of the polynomials $p_{n-1}(x)$ and $p_n(x)$ related? ($p_k(x)$ is the polynomial that solves the interpolation problem (1) for $n = k$.)

3. (Discrete Fourier Analysis) Consider the discrete inner product

   $$(f,g) = \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k)\overline{g(x_k)}, \qquad x_k = \frac{2k\pi}{2N+1},$$

   where the bar denotes complex conjugation.

(a) Show that the functions $f_j(x) = \exp(ijx)$, $j = 0, \pm 1, \pm 2, \ldots, \pm N$, (where $i = \sqrt{-1}$) are orthogonal with respect to this inner product.

(b) Let g_k be function values associated with the nodes x_k. Describe a method based on the orthogonality of the $f_j(x)$ for computing the trigonometric polynomial

$$t_N(x) = \sum_{j=-N}^{N} \alpha_j f_j(x),$$

such that

$$t_N(x_k) = g_k, \qquad k = 0, \pm 1, \pm 2, \ldots, \pm N.$$

N is assumed to be a general positive integer.

(c) Assume that $N = 2^l$ for a positive integer l. What is the "Fast Fourier Transform" algorithm, and what is the operation count when applying it to compute the trigonometric polynomial of part (b) ?

4. (Piecewise Polynomial Interpolation) A function L_Δ is in the class $\mathcal{L}_\Delta$ of "real piecewise-linear polynomials" (relative to a given partition $\Delta : a = x_0 < x_1 < \cdots < x_n = b$) if it satisfies

(a) $L_\Delta \in C[a, b]$,

(b) $L_\Delta|_{[x_i, x_{i+1}]} \in \Pi_1$, $i = 0, \ldots, n-1$,

that is, L_Δ is a continuous function that coincides with a polynomial of degree at most *one* in each subinterval.

(a) Show that $\mathcal{L}_\Delta$ is a vector space, and determine its dimension.

(b) The classical interpolation problem for $\mathcal{L}_\Delta$ is "given (x_i, f_i), $i = 0, \ldots, n$, find $L_\Delta \in \mathcal{L}_\Delta$ such that

$$L_\Delta(x_i) = f_i \,, \quad i = 0, \ldots, n \,."$$

Prove that this problem has a unique solution.

(c) Supposing that the data come from a twice continuously differentiable function, i.e., $f_i = f(x_i)$, $i = 0, \ldots, n$, ($f \in C^2[a, b]$), derive the error bound

$$\|f - L_\Delta(f)\|_\infty \le \frac{1}{8} \|f''\|_\infty \|\Delta\|^2 \,, \qquad \|\Delta\| := \max_{j=0,\ldots,n-1} |x_{j+1} - x_j| \,.$$

5. (Orthogonal Polynomials)

   (a) Given the inner product

   $$(f, g) = \int_a^b f(x)g(x)w(x)dx,$$

   where $w(x)$ is a weight function, show that the monic, orthogonal polynomials with respect to the inner product (f, g) satisfy a three term recurrence relation. Present this recurrence relation.

   (b) Give a, b and $w(x)$ for Legendre and Chebyshev polynomials.

6. (Gaussian Quadrature)

   (a) Characterize a Gaussian quadrature rule for the evaluation of $\int_a^b f(x)dx$. What are the nodes of this quadrature rule?

   (b) What are the weights?

7. (Nonlinear Equations) One wants to solve the equation $x + \ln x = 0$, whose root is near $x = 1/2$, by iteration, and one chooses between the following iteration formulas

   $$\begin{aligned} x_{n+1} &= -\ln x_n \\ x_{n+1} &= e^{-x_n} \\ x_{n+1} &= \frac{x_n + e^{-x_n}}{2} \end{aligned}$$

   (a) Which of the formulas *can* be used? Motivate your answer.

   (b) Which of the formulas *should* be used? Why?

8. (Gaussian Elimination) A matrix $H = [h_{ij}]_{i,j=1}^n \in \mathrm{R}^{n \times n}$ is called a "Hessenberg matrix" if $h_{ij} = 0$ when $i > j + 1$. Let $b \in \mathrm{R}^n$. How many arithmetic operations are necessary in order to solve

   $$Hx = b$$

   for $x \in \mathrm{R}^n$ by Gaussian elimination?

9. (QR Factorization) Let H be an upper Hessenberg matrix. Describe how it can be factored into

   $$H = QR$$

   where R is upper triangular and Q is orthogonal by

(a) Householder transformations
(b) Givens transformations
(c) Which of these approaches is to be preferred? Motivate.

10. (Iterative Methods) Let A be a nonsingular matrix and consider iterative solutions of the linear system of equations $Ax = b$.

    (a) Describe Jacobi, Gauss-Seidel and SOR iteration.
    (b) Give sufficient conditions for the Jacobi and Gauss-Seidel methods to converge.
    (c) Define "Property A" and "consistently ordered matrices." How do the concepts relate? What is their significance with respect to the SOR method?

11. (Least Squares Problems) Discuss the mathematical theory of the *linear least-squares problem*,

$$\min_{x \in R^n} \|y - Ax\|_2 , \quad A \in R^{m \times n} , \ y \in R^m \ :$$

    (a) When does this problem posses a solution?
    (b) When is the solution unique?
    (c) What are the *normal equations*, and what is their connection with this problem?
    (d) Prove that if the columns of A are linearly *dependent*, then a solution cannot be unique.

12. (Symmetric Tri-Diagonal Eigenproblem) Let B_i denote the i-th leading principal submatrix of the tridiagonal matrix

$$B = \begin{bmatrix} \delta_1 & \overline{\gamma}_2 & & \\ \gamma_2 & \delta_2 & \ddots & \\ & \ddots & \ddots & \overline{\gamma}_n \\ & & \gamma_n & \delta_n \end{bmatrix} ,$$

    and let $p_i(\mu) := \det(B_i - \mu I)$.

    (a) Derive a 3-term recurrence relation for the polynomials $p_i(\mu)$.
    (b) Describe (briefly) the usefulness of these polynomials in solving numerically the eigenproblem for B.