Numerical Analysis Qualifier

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INSTRUCTIONS: Do any 9 of the following 13 problems.

1. (Conditioning)

   Suppose that you need to code up a function subprogram to evaluate (with high relative accuracy) the function

   $$f(x) := \frac{x}{e^x - 1},$$

   for any real x.

   (a) What is the *mathematical condition number* of this function? For what ranges of x is this evaluation well conditioned, ill conditioned (with respect to relative error)?

   (b) Give a mathematically equivalent formula that is better suited for numerical computation when x is close to 0. Explain why.

2. (Polynomial Interpolation)

   (a) Given $n+1$ distinct nodes $x_0, x_1, \dots, x_n$ in a real interval $[a, b]$ and associated real numbers $f_0, f_1, \dots, f_n$, show that there is a polynomial p_n of degree at most n, such that

   $$p_n(x_i) = f_i, \qquad i = 0, 1, \dots, n. \tag{1}$$

   (b) Show that the polynomial p_n of part (a) is unique.

   (c) Suppose that the numbers f_i of part (a) are defined by $f_i = f(x_i)$, $i = 0, 1, \dots, n$, where f is a real-valued function that is differentiable arbitrarily many times on $[a, b]$. Moreover, let there be a constant M, such that

   $$\max_{a \le x \le b} \left| \frac{d^j}{dx^j} f(x) \right| \le M$$

   for all $j \ge 0$. Can it be shown without additional assumptions about the location of the x_i that $p_n(x)$ converges uniformly to $f(x)$ on $[a, b]$ as $n \to \infty$? Motivate.

3. (Polynomial Interpolation)

   Let $P_{i_0 \cdots i_k}$ denote the polynomial of degree k that interpolates to $f(x)$ at $x_{i_0}, \ldots, x_{i_k}$.

   (a) Prove that

   $$P_{i_0 \cdots i_{k+1}}(x) = \frac{(x - x_{i_0}) P_{i_1 \cdots i_{k+1}}(x) + (x_{i_{k+1}} - x) P_{i_0 \cdots i_k}(x)}{(x_{i_{k+1}} - x_{i_0})}.$$

   (b) Give the name of an algorithm that recursively uses this relationship?

4. (Discrete Fourier Analysis)

   Consider the discrete inner product

   $$(f, g) = \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k) \overline{g(x_k)}, \qquad x_k = \frac{2k\pi}{2N+1},$$

   where the bar denotes complex conjugation.

   (a) Show that the functions $f_j(x) = \exp(ijx)$, $j = 0, \pm 1, \pm 2, \ldots, \pm N$, (where $i = \sqrt{-1}$) are orthogonal with respect to this inner product.

   (b) Let g_k be function values associated with the nodes x_k. Describe a method based on the orthogonality of the $f_j(x)$ for computing the trigonometric polynomial

   $$t_N(x) = \sum_{j=-N}^{N} \alpha_j f_j(x),$$

   such that

   $$t_N(x_k) = g_k, \qquad k = 0, \pm 1, \pm 2, \ldots, \pm N.$$

   N is assumed to be a general positive integer.

   (c) Assume that $N = 2^l$ for a positive integer l. What is the "Fast Fourier Transform" algorithm, and what is the operation count when applying it to compute the trigonometric polynomial of part (b) ?

5. (Orthogonal Polynomials)

   (a) Given the inner product

   $$(f, g) = \int_a^b f(x)g(x)w(x)dx,$$

   where $w(x) > 0$ is a weight function, show that the monic, orthogonal polynomials with respect to the inner product (f, g) satisfy a three term recurrence relation.

   (b) Show that all zeros of any one of the orthogonal polynomials are simple and lie in the interval $[a, b]$.

   (c) Give a, b and $w(x)$ for Legendre and Chebyshev polynomials.

6. (Eigenvalue Problem)

   Let A be a symmetric matrix.

   (a) Show that all eigenvalues of A are real.

   (b) Show that eigenvectors associated with distinct eigenvalues are orthogonal.

7. (Eigenvalue Problem)

   (a) Define the algebraic and geometric multiplicity of an eigenvalue.

   (b) What are the algebraic and geometric multiplicities of the eigenvalues of the matrix

   $$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

8. (Matrix Representation)

   Show that any $m \times m$ matrix A admits the representation

   $$A = URU^H,$$

   where $U \in \mathbb{C}^{m \times m}$ is a unitary matrix, $R \in \mathbb{C}^{m \times m}$ is upper triangular and the superscript H denotes transposition and complex conjugation.

9. (Eigenvalue Computation)

   Let A be a real matrix of order n.

   (a) What is the power method, and how can it be applied to compute the eigenvalue of largest magnitude of A? Derive an expression for the rate of convergence.

   (b) Describe how the power method can be applied to compute the eigenvalue of smallest magnitude.

   (c) Assume that A is a real symmetric matrix of order n. Outline the QR algorithm for the computation of eigenvalues and eigenvectors of A. What is the role of the shifts in the algorithm?

10. (Condition Number)

    An alternative definition of the *matrix condition number*, which generalizes to non-square matrices, is

$$\kappa(A) := \frac{\max_{\|x\|=1} \|Ax\|}{\min_{\|x\|=1} \|Ax\|},$$

    where $\|\cdot\|$ is any vector norm.

    (a) Show that this agrees with the usual definition $\mathrm{cond}(A) = \|A\| \, \|A^{-1}\|$ when A is square and nonsingular.

    (b) For the case of the vector "2-norm", $\|\cdot\|_2$, derive an expression for this generalized condition number, $\kappa_2(A)$, in terms of the *singular values* of A.

    (c) Characterize those (not necessarily square) matrices A that satisfy $\kappa_2(A) = 1$.

11. (Householder Matrix)

    (a) Let $x \in \mathbb{R}^n$ be given. Construct a Householder matrix P, such that $Px = \alpha e_1$, where $e_1 = (1, 0, \ldots, 0)^T$ and $\alpha \in \mathbb{R}$.

    (b) Show that $\|P\|_2 = 1$.

    (c) Describe how and why Householder matrices can be applied to solve the following least-squares problem. Let the matrix $A \in \mathbb{R}^{m \times n}$ and right-hand side vector $b \in \mathbb{R}^m$ be given, and assume that $m \geq n$ and that the columns of A are linearly independent. Determine a vector $x \in \mathbb{R}^n$ such that

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2.$$

12. (Givens Matrix)

    (a) Define real Givens matrices. Why are they attractive to use for the solution of least-squares problems of the form

    $$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad m \geq n. \tag{2}$$

    (b) Let $A \in \mathbf{R}^{5 \times 4}$ be tridiagonal, i.e., all elements a_{ij} with $|i - j| > 1$ vanish. Describe how Givens matrices can be used in an efficient manner to solve the least-squares problem (2).

13. (Nonlinear Equation)

    Let f be a real-valued twice continuously differentiable function defined on an open interval on the real axis. Let f have a simple zero at the point x^* in this interval.

    (a) Describe Newton's method for computing x^*.

    (b) Show that when the initial approximate solution x_0 used in Newton's method is sufficiently close to x^*, then the iterates x_k generated by Newton's method converge to x^*.

    (c) Show that when the initial approximate solution x_0 used in Newton's method is sufficiently close to x, then the iterates x_k generated by Newton's method converge to x^* quadratically.