Numerical Analysis Qualifier

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INSTRUCTIONS: Do any 8 of the following 10 problems.

1. (Conditioning) Suppose that you need to code up a function subprogram to evaluate (with high relative accuracy) the function

$$f(x) := \frac{x}{e^{-x} - 1},$$

for any real x.

   (a) What is the *mathematical condition number* of this function? For what ranges of x is this evaluation well conditioned, ill conditioned (with respect to relative error)?

   (b) Give a mathematically equivalent formula that is better suited for numerical computation when x is close to 0. Explain why.

2. (Polynomial Interpolation)

   (a) Given $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ in a real interval $[-1, 1]$ and associated real numbers $f_0, f_1, \ldots, f_n$, show that there is a polynomial p_n of degree at most n, such that

$$p_n(x_i) = f_i, \qquad i = 0, 1, \ldots, n.$$

   (b) Show that the polynomial p_n of part (a) is unique.

   (c) Suppose that the numbers f_i of part (a) are defined by $f_i = f(x_i)$, $i = 0, 1, \ldots, n$, where f is a real-valued function that is differentiable arbitrarily many times on $[a, b]$. Moreover, let there be a constant M, such that

$$\max_{a \le x \le b} \left| \frac{d^j}{dx^j} f(x) \right| \le M$$

   for all $j \ge 0$. Prove or find a counter example to the statement: The polynomials $p_n(x)$ converges uniformly to $f(x)$ on $[a, b]$ as $n \to \infty$ for any distribution of distinct nodes x_i in $[-1, 1]$.

3. (Chebyshev Equi-Oscillation Theorem) Determine the best uniform linear approximation ($P_1 \in \Pi_1$) to the function $f(x) = e^x$ on [0,1].

4. (Composite Rules)

   (a) Given the basic Trapezoid Rule with error formula

   $$\int_a^b f(x)\,dx = \frac{(b-a)}{2}[f(a)+f(b)] - \frac{(b-a)^3}{12} f''(\xi)\,,$$

   derive the associated *composite* Trapezoid Rule with respect to a *not*-necessarily-uniform partition

   $$a = x_0 < x_1 < \cdots < x_n = b\,, \qquad h_{i+1} := x_{i+1} - x_i\,.$$

   (b) Assuming $f \in C^2[a,b]$, derive the following bound on the error:

   $$\left| \int_a^b f - T_\Delta(f) \right| \le \frac{1}{12} h^2 \|f''\|_\infty (b-a)\,, \qquad h := \max_{i=1,\dots,n} h_i\,,$$

   where $T_\Delta(f)$ denotes the composite rule.

5. (Orthogonal Polynomials) Let $d\omega(x)$ be a positive measure on $[a,b]$, and define the inner product

   $$(f,g) = \int_a^b f(x)g(x)d\omega(x).$$

   (a) Show that the family of monic orthogonal polynomials with respect to the inner product $(\cdot,\cdot)$ satisfy a three term recurrence relation.

   (b) Show that all zeros of any one of the orthogonal polynomials are simple and lie in the interval $[a,b]$.

6. (Generalized Eigenvalue Problems) Consider the *generalized symmetric definite eigenvalue problem* $Ax = \lambda Bx$, where A and B are real, square, symmetric matrices, and B is positive definite.

   (a) Show that the eigenvalues are real.

   (b) Show that the eigenvectors corresponding to distinct eigenvalues are "B-orthogonal" (i.e., $x_j^T B x_i = 0$).

(c) Construct a transformation that allows you to determine the generalized eigenvalues (and eigenvectors) by solving a *regular symmetric* eigenproblem.

7. (Matrix Representation) Show that any $m \times m$ matrix A admits the representation
$$A = URU^H,$$
where $U \in \mathbb{C}^{m\times m}$ is a unitary matrix, $R \in \mathbb{C}^{m\times m}$ is upper triangular, and the superscript H denotes transposition and complex conjugation.

8. (Singular Value Decomposition) Let the matrix $A \in \mathbb{R}^{m\times n}$ and vector $b \in \mathbb{R}^m$ be given, and assume that $m \geq n$. Consider the least-squares problem
$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2.$$
Define the *singular value decomposition* of A, and discuss how it can be used to solve this least squares problem. Do not assume that A is of full column rank.

9. (Rayleigh Quotient) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. The *Rayleigh Quotient* associated with A is the function
$$r(x) := \frac{x^T A x}{x^T x}, \quad x \in \mathbb{R}^n.$$
Note that if (λ, x) is an eigenpair of A, then it follows that $r(x) = \lambda$.

   (a) Show that the Rayleigh Quotient is stationary on the eigenvectors of A, that is, if x is an eigenvector of A, then $\nabla r(x) = 0$ (where ∇ denotes the gradient).

   (b) Use this result to show that if y is an approximation to the eigenvector x, with associated eigenvalue λ, then
$$|\lambda - r(y)| = O\Big(\|x - y\|_2^2\Big).$$

10. (Nonlinear Equation) Let $f \in C^\infty$ have a zero at x_*.

   (a) Show that when the initial approximate solution x_0 is sufficiently close to x_* and x_* is a simple zero, then the iterates x_k determined by Newton's method converge to x_* at least quadratically.

   (b) What is the rate of convergence of Newton's method when x_* is a zero of multiplicity 2?