Numerical Analysis Qualifier

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August, 2007

INSTRUCTIONS: Do any (and only) 8 of the following 10 problems.

1. (Rounding-Error Analysis) Consider a computer with four digits precision where any real number \( x = 1.d_1d_2d_3d_4d_5d_6 \cdots \times 10^E \) is represented by its floating point number \( f(x) = 1.d_1d_2d_3d_4 \times 10^E \).

   (a) Determine the value \( 100.0 \oplus 0.001 \), where \( \oplus \) represents the addition implemented on this computer.

   (b) The exact solution of

   \[
   \begin{bmatrix}
   0.001 & 100.0 \\
   100.0 & 100.0
   \end{bmatrix}
   \begin{bmatrix}
   x_1 \\
   x_2
   \end{bmatrix} = \begin{bmatrix}
   100.0 \\
   0.0
   \end{bmatrix}
   \tag{1}
   \]

   is \([x_1, x_2] = [-1.0, 1.0]\) (up to the machine epsilon).

   Solve (1) by implementing Gaussian Elimination without pivoting on the above mentioned computer. What is the error of the solution in Euclidean norm? What is its relative error?

   Solve (1) by implementing Gaussian Elimination with pivoting.

2. (Singular Value Decomposition) Given a matrix \( A \in \mathbb{R}^{n \times n} \),

   (a) Describe the Singular Value Decomposition of \( A \).

   (b) Let \( \sigma_1 \) be the largest singular value of \( A \). Prove that \( \| A \|_2 = \sigma_1 \), where \( \| A \|_2 \) is defined by

   \[
   \| A \|_2 = \max_{x \neq 0} \frac{\| Ax \|_2}{\| x \|_2}.
   \]

3. (Rayleigh Quotient) Let \( A \in \mathbb{R}^{n \times n} \) be symmetric. The Rayleigh Quotient associated with \( A \) is the function defined by

   \[
   r(x) := \frac{x^T Ax}{x^T x}, \quad x \in \mathbb{R}^n.
   \]

   (a) Prove that \( \| Ax - r(x)x \|_2 = \min_{\mu \in \mathbb{R}} \| Ax - \mu x \|_2 \).
(b) Assume that $A$ is positive definite. Denote its minimum and maximum eigenvalues by $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$. Prove that

$$
\lambda_{\text{min}} = \min_{x \neq 0} \frac{x^T Ax}{x^T x}, \quad \lambda_{\text{max}} = \max_{x \neq 0} \frac{x^T Ax}{x^T x}.
$$

4. (Arnoldi Iteration) Given a matrix $A \in \mathbb{R}^{m \times m}$ and a column vector $q_1 \in \mathbb{R}^m$ with $\|q_1\|_2 = 1$,

(a) Present an algorithm which generates a sequence of orthonormal vectors $q_1, q_2, \ldots$ and an upper Hessenberg matrix $H_n \in \mathbb{R}^{(n+1) \times n}$ (where $n < m$) such that

$$
A \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} = \begin{bmatrix} q_1 & \cdots & q_{n+1} \end{bmatrix} \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ h_{21} & \cdots & h_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & h_{n+1,n} \end{bmatrix}.
$$

(b) Prove that $\langle q_1, Aq_1, \ldots, A^n q_1 \rangle = \langle q_1, q_2, \ldots, q_{n+1} \rangle$.

5. (GMRES Algorithm) Given a matrix $A \in \mathbb{R}^{m \times m}$ and a column vector $b \in \mathbb{R}^m$, using the result in Question 5, present an iteration which solves

$$
\min_{x \in \{b, Ab, \ldots, A^{n-1} b\}} \|Ax - b\|_2,
$$

at the $n$-th iteration.

6. (Polynomial Interpolation) Given a set of points $(x_i, f_i)$, $i = 0, 1, \ldots, n$, where $x_0 < x_1 < \ldots < x_n$, and $f_i = f(x_i)$, let $p(x)$ be the interpolation polynomial of degree less or equal than $n$, such that $p(x_i) = f_i$, $i = 0, 1, \ldots, n$. Assume that $f^{(n+1)}(x)$ is continuous. Prove that for any $x \in [x_0, x_n]$, there exists $\xi \in [x_0, x_n]$ such that

$$
f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - x_0)(x - x_1)\ldots(x - x_n).
$$

7. (Trigonometric Interpolation) Given a sequence of points $(x_k, f_k)$, $k = 0, 1, \ldots, N-1$, where $x_k = 2\pi k/N$,

(a) Prove that there exists a unique phase polynomial of the form

$$
p(x) = \sum_{j=0}^{N-1} \beta_j e^{ijx},
$$

where $i$ denotes the imaginary unit, such that $p(x_k) = f_k$, for $k = 0, 1, \ldots, N-1$. 

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(b) Prove that the coefficients $\beta_j$ in (a) are determined by

$$\beta_j = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-2\pi ijk/N}, \quad j = 0, 1, \ldots, N - 1.$$  

8. (Peano Kernel Theorem) The Peano Kernel Theorem states that if a functional $R(f)$ satisfies $R(P) = 0$ for all polynomials $P$ of degree less or equal than $n$, then for all functions $f \in C^{n+1}[a, b]$,

$$R(f) = \int_a^b f^{(n+1)}(t) K(t) dt,$$

where $K(t) = R_{x(t)}[(x-t)^n]/n!$ and $R_{x(t)}[(x-t)^n]$ represents the application of $R$ on $(x-t)_+^n$ considered as a function of $x$.

Using the Peano Kernel Theorem, prove that for any $f(x) \in C^2[a, b]$, there exists $\xi \in [a, b]$, such that

$$\frac{b-a}{2} \left( f(a) + f(b) \right) - \int_a^b f(x) dx = \frac{(b-a)^3}{12} f''(\xi).$$

(Hint: Take $R(f) = \frac{b-a}{2} (f(a) + f(b)) - \int_a^b f(x) dx$ in the proof.)

9. (Gauss Quadrature) Let $p_j(x) \in \{ p \mid p(x) = x^j + a_1 x^{j-1} + \cdots + a_j \}$, $j = 0, 1, \ldots, n$, be a set of orthogonal polynomials with respect to the inner product

$$(f, g) = \int_a^b \omega(x) f(x) g(x) dx,$$

where $\omega(x)$ is a nonnegative smooth function. Let $x_1, x_2, \ldots, x_n$ be the distinct roots of $p_n(x)$, and $w_1, w_2, \ldots, w_n$ be determined by

$$\begin{bmatrix}
p_0(x_1) & p_0(x_2) & \cdots & p_0(x_n) 
p_1(x_1) & p_1(x_2) & \cdots & p_1(x_n) 
\vdots & \vdots & \ddots & \vdots 
p_{n-1}(x_1) & p_{n-1}(x_2) & \cdots & p_{n-1}(x_n)
\end{bmatrix} \begin{bmatrix}
w_1 
w_2 
\vdots 
w_n
\end{bmatrix} = \begin{bmatrix}
(p_0, p_0) 
(p_0, p_1) 
\vdots 
(p_0, p_{n-1})
\end{bmatrix}.$$

Prove that

$$\int_a^b \omega(x) p(x) dx = \sum_{i=1}^{n} w_i p(x_i)$$

hold for all polynomials $p(x)$ of degree less or equal than $2n - 1$.

10. (Nonlinear Equations) Describe the Newton’s iteration for solving

$$e^{-x} = x.$$  

Prove that it is convergent starting from $x_0 = 1$, and determine its rate of convergence.