QUALIFYING EXAM IN ALGEBRA

Instructions:

a) Do all questions in part I.

b) Do as many questions as you can in part II, but you must attempt at least four questions.

Policy on Misprints

The committee tries to proofread the exams as carefully as possible. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation of your solution. In such cases do not interpret a problem in such a way that it becomes trivial.
1. Consider the following groups:
\[ \mathbb{C}^* \text{ the multiplicative group of nonzero complex numbers.} \]
\[ \mathbb{R}^+ \text{ the multiplicative group of positive real numbers.} \]
\[ N = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R}, a^2 + b^2 \neq 0 \right\} \text{ under multiplication.} \]
\[ U = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\} \text{ under multiplication} \]

(a) Prove \( \mathbb{C}^* \approx N \).

(b) Prove \( U \) is a normal subgroup of \( N \) with \( N/U \approx \mathbb{R}^+ \).

2. Let \( G \) be a group. Assume that there exists an integer \( n > 1 \) with \( (xy)^n = x^n y^n \) for all \( x, y \in G \).

Set \( G_n = \{ x \in G | x^n = 1 \} \)
\[ G^n = \{ x^n | x \in G \}. \]

Prove \( G_n \) and \( G^n \) are normal subgroups of \( G \) with \( [G: G_n] = |G^n| \).

3. An element \( a \) of a ring \( R \) is said to be nilpotent if \( a^n = 0 \) for some integer \( n > 0 \). Describe explicitly the ideals \( I \) of the ring \( \mathbb{Z} \) of integers such that \( \mathbb{Z}/I \) has no nonzero nilpotent elements.

4. Let \( X \) be a nonempty set and let \( R = \{ f | f: X \to \mathbb{Z} \} \). Given \( f, g \in R \) define \( f + g \) and \( f \cdot g \) by
\[ (f + g)(x) = f(x) + g(x) \]
\[ (f \cdot g)(x) = f(x) \cdot g(x) \text{ for all } x \in X. \]

(a) Prove that \( R \) is a commutative ring with identity element.

(b) Prove that for each nonempty subset \( Y \) of \( X \),
\[ \{ f \in R | f(y) = 0 \ \forall y \in Y \} \]
is an ideal of \( R \).
Part I (contd.)

5. Let $K$ be an extension field of a field $k$ and let $a, b$ be elements of $K$ both algebraic over $k$ with $\text{Irr}(k, a) = \text{Irr}(k, b)$. Prove there exists a $k$-isomorphism $\sigma$ of $k(a)$ onto $k(b)$ with $\sigma(a) = b$. \hfill $\Box$
PART II

1. Give a complete list (up to isomorphism) of all groups $G$ with $|G| = 7^2 \cdot 5^2 = 1225$.

2. Let $G$ be a finite group with proper subgroup $H$. Prove that

$$G \neq \bigcup_{x \in G} xHx^{-1}.$$ 

3. Let $R$ be a commutative ring and let $x$ be a nonzero element of $R$.

   (a) Prove there exists an ideal $I$ of $R$ maximal with respect to $x \notin I$.

   (b) Prove that the principal ideal of $\overline{R} = R/I$ generated by $\overline{x} = x + I$ is contained in every nonzero ideal of $R/I$.

4. Outline the proof of the following theorem:

   If $A$ is a Gaussian integral domain, then $A[x]$ is also a Gaussian integral domain.

   (Note: Gaussian integral domain $\equiv$ unique factorization domain).

5. Let $K$ be a finite normal separable field extension of a field $k$ with $k \neq K$. Assume $a^3 \in k$ for all $a \in K$. Prove that $G(K/k)$ is cyclic.

6. Let $K, F, L$ be fields. Prove that if $K$ is an algebraic extension of $F$ and $F$ is an algebraic extension of $L$, then $K$ is an algebraic extension of $L$. 