QUALIFYING EXAM IN ALGEBRA

Instructions:

a) Do all questions in part I.

b) Do as many questions as you can in parts II, III, IV, but you must attempt one from each part.

Policy on Misprints

The committee tries to proofread the exams as carefully as possible. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation of your solution. In such cases do not interpret a problem in such a way that it becomes trivial.
PART I.

1. Let $A$ and $B$ be subgroups of a group $G$ with $A$ a normal subgroup. Prove that $AB = BA$ and that $AB$ is a subgroup of $G$.

2. Let $(R, +, \cdot)$ be a ring with identity $1$. Define $\oplus$ and $\odot$ on $R$ by $a \oplus b = a + b + 1$ and $a \odot b = (a \cdot b) + a + b$.

   (i) What is the identity for $R$ under $\oplus$?

   (ii) What is the identity for $R$ under $\odot$?

   (iii) Exhibit a one-to-one and onto map $f: R \rightarrow R$ such that

   $f(a + b) = f(a) \oplus f(b)$

   $f(a \cdot b) = f(a) \odot f(b)$

   and prove that $(R, \oplus, \odot)$ is a ring.

3. Let $H$ be a normal subgroup of a finite group $G$ and let $f: G \rightarrow H$ be a homomorphism such that $f(h) = h$ for all $h \in H$. Show there is a subgroup $K$ of $G$ such that $G$ is the (internal) direct product of $H$ and $K$.

4. (a) Define what is meant by the characteristic of a field.

   (b) Define what is meant by a prime field.

   (c) List all prime fields up to isomorphism and prove that your list is complete.

5. (a) Let $\{I_i\}_{i=1}^{n}$ be a finite collection of non-zero ideals of an integral domain $R$. Prove that $\bigcap_{i=1}^{n} I_i \neq [0]$.

   (b) If $\{I_{\alpha}\}_{\alpha \in \Gamma}$ is an arbitrary collection of non-zero ideals of an integral domain $R$, is $\bigcap_{\alpha \in \Gamma} I_{\alpha} \neq [0]$?
PART II.

1. (a) State the fundamental theorem for finite abelian groups.
   
   (b) How many nonisomorphic abelian groups are there of order 360? Why?
   
   (c) If $A$, $B$ and $C$ are finite abelian groups with $A \oplus B \cong A \oplus C$, is $B \cong C$? Either prove or disprove.

2. (a) Let $\sigma$ be a permutation on a set $X$ that fixes $i \in X$ and does not fix $j \in X$. Show that $(\sigma)$, the subgroup generated by $\sigma$, is not a normal subgroup of $S_X$, the symmetric group on $X$.
   
   (b) Exhibit a subgroup of $S_4$ that is not normal.

3. Let $G$ be a group of order $pq$ where $p$ and $q$ are primes with $p < q$. Prove that if $p$ does not divide $q - 1$, then $G$ is cyclic.
PART III.

1. Let $R$ be a ring with identity element $1$ and let $I$ and $J$ be two-sided ideals of $R$ with $R = I + J$ and $I \cap J = \{0\}$. Prove:
   
   (i) there exist unique elements $x \in I$, $y \in J$ with $1 = x + y$.
   
   (ii) $x \cdot x = x$, $y \cdot y = y$, $x \cdot y = y \cdot x = 0$.
   
   (iii) $I = Rx$, $J = Ry$.

2. Let $A$ be an integral domain having the property that for any two ideals $I$ and $J$ of $A$ either $I \subseteq J$ or $J \subseteq I$. Prove that an element of $A$ is irreducible if and only if it is prime.

3. Let $R$ be a commutative ring with identity $1$. Let $a \in R$ with $a^n \neq 0$ for all $n \geq 1$.

   (i) Prove there exists an ideal $P$ of $R$ maximal with respect to $P \cap \{1, a, a^2, \ldots\} = \emptyset$.

   (ii) Prove that $P$ is a prime ideal of $R$. 
PART IV.

1. Let \( K \) be an extension field of \( k \). Prove that \( K \) is a finite normal extension of \( k \) if and only if \( K \) is the splitting field of some polynomial over \( k \).

2. Let \( K, L \) and \( M \) be subfields of a field \( F \) with \( L \cap M = K \). Suppose \([M: K] = m < \infty \) and \([L: K] = n < \infty \). Prove that \([L \lor M: K] \leq m \cdot n \) where \( L \lor M \) denotes the smallest subfield of \( F \) containing both \( L \) and \( M \).

3. Prove that no finite field is algebraically closed.