Work ALL problems in this part.

PART A

A 1. a) Prove that if the power series \( \sum_{n=0}^{\infty} a_n z^n \) converges at \( z_0 \neq 0 \) then it converges absolutely and uniformly on every compact subset of the disc \( \{ z : |z| < |z_0| \} \).

b) Prove that the sum of the series in part a) represents an analytic function on the disc \( \{ z : |z| < |z_0| \} \).

c) Prove that there exists at least one singular point of the function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) on its circle of convergence.

d) Find the radius of convergence of the series \( \sum_{n=1}^{\infty} \frac{z^n}{n} \).

e) For which points on its circle of convergence does the series in part d) converge?

f) Does your answer in part e) contradict the result of part c)? Explain.

g) Find the function \( f(z) \) represented by \( \sum_{n=1}^{\infty} \frac{z^n}{n^2} \) within its circle of convergence (i.e. find the sum of the series).

A 2. a) Describe the behaviour of an analytic function \( f(z) \) in a neighborhood of

i) a pole.

ii) a removable singular point.

iii) an isolated essential singular point.

b) Show that the function \( e^z \) assumes every complex value \( c, \ (c \neq 0), \) in every neighborhood of \( \infty \).

A 3. Evaluate \( \frac{1}{2\pi i} \oint_{|z|=2} \frac{ze^z dz}{(z-1)^2(z+3)} \).

A 4. Find the Laurent expansion for \( f(z) = \frac{z}{(z-1)(z+3)} \) valid in the annulus \( 1 < |z| < 3 \).

A 5. a) State the Cauchy Integral Theorem.

b) Derive the Cauchy Integral Formula. (You may use the result of part a) of this problem).
PART B

Work one of the following problems.

B 1. a) Let the functions $f(z)$ and $g(z)$ be analytic at $a$. If $f(a) \neq 0$, $g(a) = 0$, $g'(a) \neq 0$, show that the residue of $\frac{f(z)}{g(z)}$ at $a$ is $\frac{f(a)}{g'(a)}$.

b) Find the residue of $\tan z$ at $\frac{\pi}{2}$.

B 2. If $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$, find the Taylor series expansion of $f(z)$ about 0.

PART C

Work one of the following problems.

C 1. Evaluate $\int_{\infty}^{\infty} \cos \frac{x}{1+x^2} \, dx$.

C 2. Prove that if $f(z)$ has a simple pole at $a$ with residue $b$, and if $C$ is the arc, $\theta_1 \leq \arg(z-a) \leq \theta_2$, of the circle $|z-a| = r$, then

$$\lim_{r \to 0} \int_{C} f(z)\,dz = ib(\theta_2 - \theta_1).$$

PART D

DEFINITION: An analytic function is said to be univalent on a domain $G$ provided that $f(z_1) = f(z_2)$ if and only if $z_1 = z_2$ for all $z_1$ and $z_2$ in $G$.

D 1. Find a M"obius (bilinear) transformation which maps the circle $|z| \leq 1$ onto $|w - 1| \leq 1$ and takes the points 0 and 1 in the $z$ plane into $1/2$ and 0 respectively in the $w$ plane. Is the transformation uniquely determined by the data?

D 2. Suppose $f(z)$ is analytic in a simply connected region $G$ and on its boundary $\gamma$. Prove that if $f(z)$ is univalent on $\gamma$, then $f(z)$ is univalent in $G$.

D 3. Given $f(z)$ an analytic function on a simply connect region $G$, prove that the condition $f'(z) \neq 0$ on $G$ is necessary but not sufficient for the univalence of $f(z)$ on $G$. 