Qualifying Examination in Algebra

March 29, 1971
Instructions:

(1) Do all the problems marked with an asterisk.
(2) Do a total of eight (8) problems subject to
   a) At least two from each of the Sections I, II, III.
   b) At most three from any one of the Sections I, II, III.

Policy on Misprints

The committee tries to proofread the exams as carefully as possible. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation of your solution. In such cases do not interpret a problem in such a way that it becomes trivial.
I. GROUP THEORY

1. Let $G$ be a group of order $p^2 q$, $p$ and $q$ distinct primes, such that $p \neq 1 \mod q$. Show that $G$ has a normal abelian subgroup.

2*(i) State the Fundamental Theorem for finitely generated abelian groups. (Also referred to as the Basis Theorem for finitely generated abelian groups.)

(ii) Show if $G$ is a finite cyclic group and $H_1, H_2$ subgroups of $G$ such that $G/H_1 \cong G/H_2$, then $H_1 \approx H_2$.

3. Let $G$ be a finite group and $H$ a normal subgroup of $G$. If $S$ is a Sylow $p$-subgroup of $G$, $p$ a prime, and $S$ is a normal subgroup of $H$, then $S$ is a normal subgroup of $G$.

4. Let $H$ be a proper normal subgroup of $S_n$, $S_n$ the symmetric group of degree $n$, such that $H$ contains a 5-cycle, then $H = A_n$, $A_n$ the alternating group of degree $n$.

5. Let $0 \neq n \in \mathbb{Z}$, $\mathbb{Z}$ the additive group of integers. Exhibit a composition series for $\mathbb{Z}/(n)$, the integers modulo $n$ as an abelian group, and prove it is a composition series.
II. RING THEORY

1. Let \( K \) be a field and \( R \) the ring of all \( n \times n \) upper triangular matrices over \( K \) (i.e. having zeroes below the main diagonal), where \( n \) is a fixed positive integer.

Show: (i) For each positive integer \( k, 1 \leq k \leq n \), \( \mathcal{M}_k \) the set of all matrices \((A_{ij})\) in \( R \) for which \( A_{kk} = 0 \), is a maximal right ideal of \( R \).

(ii) The Jacobson radical of \( R \) is the set of all strictly upper triangular matrices.

2. Let \( F[x,y] \) be the polynomial ring in indeterminates \( x \) and \( y \) over the field \( F \). Prove or disprove each of the following.

(i) \( F[x,y] \) is a Gaussian domain (unique factorization domain).

(ii) \( F[x,y] \) is a Euclidean domain.

(iii) Every nonzero prime ideal of \( F[x,y] \) is a maximal ideal.

3. Describe completely the structure of a finite ring with identity which has no non-zero nilpotent elements.

4*. Let \( D \) be an integral domain (i.e. a commutative ring with identity and no zero divisors) and \( F \) its field of quotients, with \( D \subseteq F \). Show that if \( F \) has the property that for \( 0 \neq x \in F \), either \( x \in D \) or \( x^{-1} \in D \), then for \( a, b \in D, a \neq 0, b \neq 0 \), either \( a \) divides \( b \) or \( b \) divides \( a \).

5. Let \( R \) be a ring and \( M \) a left \( R \)-module. Show the following statements are equivalent.

(i) Every submodule of \( M \) is finitely generated.

(ii) For every ascending sequence \( M_1 \subseteq M_2 \subseteq \cdots \) of submodules of \( M \), there is an index \( N \) such that \( M_\infty = M_\infty^{N+1} = \cdots \).
III. FIELD THEORY

1. Let $K$ be a field and $f(x)$ a polynomial in $K[x]$. Prove there exists a field $L$ containing $K$ as a subfield such that $f(x)$ has a root in $L$.

2. Given a tower of fields $K \subseteq L \subseteq P$, prove that if $L$ is algebraic over $K$ and $P$ is algebraic over $L$, then $P$ is algebraic over $K$.

3. State the Fundamental Theorem of Galois Theory and define all relevant terms used.

4. Show that every finite field $K$ is a splitting field of a polynomial in $k[x]$, where $k$ is the prime subfield of $K$.

5. Definition: A field $F$ is a normal extension of a subfield $K$, if every irreducible polynomial $f(x)$ in $K[x]$ which has a root in $F$ splits in $F$.

   Let $K$ be a subfield of $L$. Show there is a subfield $F$ of $L$ containing $K$ such that $F$ is maximal with respect to the property that $F$ is a normal extension of $K$. 
Preliminary Examination - Complex Variables 30 March 1971

There are seven problems on the examination. Do all of your work on separate paper. Write on one side only and begin each problem on a separate page. In the examination the words holomorphic, analytic, and regular are synonymous.

1. a. Define the exponential function $e^z$.

   b. Using the definition given in part (a.) show that for any complex number $c$ there exists a sequence $\{z_n\}$, $|z_n| \to \infty$ such that $\lim_{n \to \infty} e^{z_n} = c$.

2. Let $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $1$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\{z : |z| < 1\}$.

   a. Define what is meant by a singular point of $f$ on the circle $\{z : |z| = 1\}$.  

   b. Show that there exists at least one singular point of $f$ on the circle $\{z : |z| = 1\}$.

3. a. State the Cauchy-Goursat Theorem for triangles.

   b. Prove the theorem stated in part (a.).

4. If $D$ is a domain and $f$ is non-constant and holomorphic on $D$ then $f$ is an open mapping, i.e., $f$ maps open sets onto open sets. Assuming this fact

   a. deduce the maximum principle (maximum modulus principle)

   b. deduce the "fundamental theorem of algebra"

5. Let $\{f_n(z)\}$ be a sequence of functions, each holomorphic on a domain $D$. Assume that the sequence converges locally uniformly on $D$, i.e., the sequence converges uniformly on every compact subset of $D$.

   a. Show that a limit function exists and is holomorphic on $D$.

   b. Show that, if in addition each function $f_n$ is one-to-one on $D$, then either the limiting function is constant or one-to-one on $D$. 
6. a. State the Riemann Mapping Theorem and explain the terms which you use in your statement.

   b. Either give a conformal mapping of the complex plane onto the unit disc or show that no such mapping exists.

7. Let \( f(z) = \frac{z}{(1-z)^2} \).

   a. Show that \( f \) is one-to-one and holomorphic on the unit disc \( D \).

   b. Find the image \( f[D] \) and make a sketch of the mapping.
Instructions.  a) Do at least one problem from each section.
    b) Do no more than six problems.

1.1.  (a) State Banach's contraction mapping theorem.

    (b) Let \( K: [a, b] \times [a, b] \to \mathbb{R} \) be continuous. Under what conditions on \( K \)
    will there exist a continuous \( f: [a, b] \to \mathbb{R} \) such that
    \[
    f(x) = \int_a^b K(x, y)f(y) \, dy \quad \text{for all } x \in [a, b].
    \]
    Explain your reasons.

1.2.  (a) State a definition of compactness in a metric space.

    (b) State at least two other conditions equivalent to the definition in (a).

    (c) Prove that the definition in (a) is equivalent to one of the conditions
        in (b).

1.3.  (a) State the Baire category theorem.

    (b) Prove one of the following:
        (i) Baire category theorem
        (ii) Every infinite dimensional Banach space has uncountable Hamel
            dimension.

2.1.  Let \( 0 < a < 1 \) and define \( f(x) = \frac{1}{a} \) for \( 0 < x \leq 1 \). Prove that \( f \) is Lebesgue
    integrable on \( (0, 1] \) and that \( \int_{(0,1]} x^a f \, dm = \frac{1}{1-a} \), where \( m \) denotes Lebesgue
    measure.

2.2.  Does it follow from \( \int_0^1 x^n f(x) \, dx = 0 \) for \( n = 0, 1, 2, \ldots \), that \( f(x) = 0 \) in
    \([0,1]\)? Consider this question in the different spaces of functions:
    \( \mathcal{C}[0,1], L^p[0,1], L^\infty[0,1] \). Justify your answer.

2.3.  Let \( E \) be a Lebesgue measurable subset of the unit square in the plane. Suppose
    there exists a positive number \( \delta \) such that \( m(E_x) > \delta \) for all \( x \in [0,1] \). Prove
    that there exists a positive number \( \Delta \) and a set \( C \) of positive measure such that
    \( m(E^y) > \Delta \) for all \( y \in C \). (Note: \( E_x = \{ y: (x,y) \in E \}, E^y = \{ x: (x,y) \in E \} \).
3.1. (a) State the dominated convergence theorem.
    (b) State Egoroff's theorem.
    (c) Prove one of the theorems stated in (a) and (b).

3.2. Let \( A \) be a \( \sigma \)-algebra of subsets of \( X \).
    
    (a) What is meant by a measurable, extended real-valued function on \( X \)?
    
    (b) Let \( \{ f_n \}_{n=1}^{\infty} \) be a sequence of measurable functions on \( X \). Prove that
        \( \lim \sup_{n} f \) is measurable.

3.3. Let \((X,A)\) be a measurable space and assume \( f \) is a measurable function on
    \( X \). If \( B \) is a Borel set, prove that \( f^{-1}(B) \) is measurable.

4.1. Let \( H \) be a Hilbert space with norm \( \| \cdot \| \) and inner product \( (\cdot,\cdot) \), and let \( V \)
    be any finite-dimensional subspace of \( H \). Given any \( f \in H \),
    
    a) show that there exists a unique \( \tilde{g} \) of best approximation to \( f \) in \( V \), i.e.,
        \[ \| f - \tilde{g} \| = \inf_{w \in V} \| f - w \|. \] (Hint: Use the parallelogram law).
    
    b) show that \( (f-\tilde{g}, w) = 0 \) \( \forall w \in V \).

4.2. Let \( H \) be a Hilbert space with norm \( \| \cdot \| \) and inner product \( (\cdot,\cdot) \), and let
    \( \{\phi_j\} \) be any orthonormal sequence in \( H \).
    
    a) Show that all elements of the form \( \sum_{j=1}^{m} a_j \phi_j \) ( \( m \) arbitrary) is dense in \( H \)
        if and only if, for any \( f \in H \), \( f = \sum_{j} c_j \phi_j \) where \( c_j = (f, \phi_j) \).
    
    b) For any \( f \in L_2[0,1] \), let \( \tilde{p}_n(x) \) be the unique polynomial of degree \( \leq n \)
        of best approximation of \( f \) in the \( L_2 \)-sense on \([0,1]\). Show that
        \[ \| f - \tilde{p}_n \|_{L_2[0,1]} \] is a nonincreasing sequence converging to zero.
4.3. If $B$ is a compact Hausdorff topological space and if $C(B)$ denotes the collection of all real-valued continuous functions on $B$, then $C(B)$ is a Banach space with respect to the norm: $\|f\| = \sup_{x \in B} |f(x)|$. Let $V$ be any finite-dimensional subspace of $C(B)$, and define

$$\rho_v(f) = \inf_{g \in V} \|f-g\|$$

for any $f \in C(B)$. Let $\mathcal{F}$ be the collection of all continuous linear functionals $L$ on $C(B)$ with $L(g) = 0$ for all $g \in V$ and $\|L\| \leq 1$.

a) Show first that $|L(f)| \leq \rho_v(f)$ for all $L \in \mathcal{F}$.

b) Show that $\sup_{L \in \mathcal{F}} |L(f)| = \rho_v(f)$.

(Hint: Apply the Hahn-Banach Theorem.)
Comprehensive examination

TOPOLOGY

1. Answer exactly 12 question. Select

   3 from part I
   3 from part II
   3 from part III
   3 from part IV

2. When you are required to prove a particular statement, give as many details as time permits. In any case, be sure to give the main steps of the proof. In case of examples, an accurate description is sufficient.

3. The examination was checked carefully for misprints; if you think there is a mistake, do not interpret the question so as to make it trivial.

4. In some textbooks normal, regular spaces are defined to be Hausdorff. In this exam, do not take any of these terms to imply Hausdorff. So, for example, if a problem requires you to prove a space is normal, then you do not have to prove the space is Hausdorff.
PROVE OR DISPROVE THE FOLLOWING STATEMENTS

I. 1. Every compact Hausdorff space is normal.

2. A space $X$ is Hausdorff iff $\{ (x,x) : x \in X \}$ is closed in $XXX$.

3. Let $X$ be a set with a linear order. The space $X$ with the order topology is regular.

4. State and prove three equivalent definitions of continuity.

5. (Def: A space is locally compact iff each point has a neighborhood base of compact sets.)

A Hausdorff space is locally compact iff each point has a compact neighborhood.

6. (Def: A set $C$ is a cozero set in a space $X$ if there exists a real-valued continuous function $f$ on $X$ such that $C=f^{-1}(\mathbb{R}/\{0\})$)

$X$ is completely regular iff $\{C : C$ is a cozero set in $X\}$ forms a base for the topology

II. 1. Let $f$ be a real-valued function whose domain is a compact Hausdorff space. If $f$ is upper semicontinuous, then $f$ attains its supremum.

2. Let $f$ be a continuous function from a compact metric space $Y$ onto a metric space $Z$. Then $f$ is uniformly continuous.

3. Let $X$ and $Y$ be Hausdorff and $Y$ be compact. Then $f : X \rightarrow Y$ is continuous iff graph $G(f)$ is closed in $X \times Y$.

4. (Def: $S$ is said to be $C^*$-embedded in $X$ if every bounded real-valued continuous function on $S$ can be extended to a bounded real-valued continuous function on $X$.)

Let $\beta X$ be the Stone-Cech compactification of $X$. Then $S$ is $C^*$-embedded in $X$ iff $\overline{\text{cl}_X S} = \beta S$.

5. $X$ is Hausdorff and locally compact iff $X$ is open in any Hausdorff compactification.

6. The irrationals are separable.
III. 1. Let $Y$ be compact and let $p_z$ be the projection of $Y \times \mathbb{Z}$ onto $\mathbb{Z}$. Then $p_z$ is a closed map.

2. The countable product of separable non-empty spaces is separable.

3. The closed continuous image of a normal space is normal.

4. (Def: A decomposition $\mathcal{D}$ of a space $X$ is upper semi-continuous iff whenever $F \in \mathcal{D}$ and $F \subseteq U$ where $U$ is open, then there is an open set $V$ which is a union of elements of $\mathcal{D}$ and $F \subseteq V \subseteq U$.)

The natural map $p : X \to X/\mathcal{D}$ is closed iff the decomposition is upper semi-continuous.

5. The product of normal spaces is normal.

6. The product of Lindelöf spaces is Lindelöf.

IV. 1. Using connectedness, prove $I$ is not homeomorphic to $I \times I$.

2. The intersection of a nest of non-empty continua in a Hausdorff space is a non-empty continuum.

3. (Def: Let $X$ be dense in $T$ and let $F$ be a filterbase on $X$. Then $F$ is said to converge to the limit $p \in T$ if every neighborhood (in $T$) of $p$ contains a member of $F$.)

Let $T$ be Hausdorff. Two different filterbases on $X$ cannot have a common limit in $T$.

4. Suppose the metric space $(X, d)$ is complete and the metric space $(Y, p)$ is homeomorphic to $(X, d)$. Then the metric $p$ is complete.

5. For what classes of spaces does the following statement hold?

$x \in \text{cl} S$ iff there exists a sequence in $S$ converging to $x$.

Justify your answer.

6. Every compact connected Hausdorff space is locally connected.