QUALIFYING EXAM

April 3, 1972
Instructions:

(1) Do all the problems marked with an asterisk.

(2) Do a total of eight (8) problems subject to
   a) At least two from each of the Sections I, II, III.

Policy on Misprints

The committee tries to proofread the exams as carefully as possible. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation of your solution. In such cases do not interpret a problem in such a way that it becomes trivial.
I. GROUP THEORY

1. (a) State the class equation (conjugate class equation) for a finite group.

   (b) Show that a finite p-group, p a prime, has a non-trivial center.

2. Let $K, H$ be subgroups of a group $G$ such that $K \subseteq H$. State the hypotheses of an isomorphism theorem for groups whose conclusion is

   \[ \frac{G/K}{(H/K)} \cong G/H \]

   and prove it!

3. Show that a group of order 56 is not simple. Hint: Use the Sylow Theorems.

4. Let $S_p$ be the symmetric group on $p$ elements, $p$ a prime.

   (a) Show that $S_p$ has exactly $(p-1)!$ elements of order $p$.

   (b) Show that each Sylow $p$-subgroup has $(p-1)$ elements of order $p$.

   (c) Show that if $n$ is the number of Sylow $p$-subgroups of $S_p$, then

       \[ n(p-1) = (p-1)! \]

   (d) Show that $(p-1)! \equiv -1 \pmod{p}$.

5. (a) Show that a finite abelian group is a direct sum of a finite number of $p$-groups.

   (b) State the fundamental theorem for finite abelian groups.
II. RING THEORY

1. Let $R$ be a commutative ring. An ideal $Q$ of $R$ is called irreducible, if whenever $Q = A \cap B$, for ideals $A$, $B$ of $R$, then $A = Q$ or $B = Q$. Show that a prime ideal of $R$ is irreducible.

2. Let $R$ be a ring. Show there is an ideal $P$ which consists of zero-divisors only, and if $P \subseteq Q$, $Q$ an ideal of $R$, then $Q$ has a non-zero divisor.

3. Let $R$ be a principal ideal domain (i.e., a commutative ring with identity, which has no zero divisors and every ideal is principal). If $b, c \in R$ and $a \in R$ is the greatest common divisor of $b$ and $c$, then state and prove the relationship between the principal ideals generated by $b, c$ and $a$ respectively.

4. Let $Q$ denote the field of rational numbers and $p$ a prime. Denote $Q_p = \{ \frac{m}{n} \mid (n, p) = 1, m, n \in \mathbb{Z} \}$

(a) Show $Q_p$ is an integral domain.

(b) Show that $Q$ is the field of quotients of $Q_p$.

(c) Show that $\{ \frac{mp}{n} \mid (n, p) = 1, m, n \in \mathbb{Z} \}$ is the unique maximal ideal of $Q_p$.

(d) Show if $x \in Q$, then either $x \in Q_p$ or $x^{-1} \in Q_p$.

5. Let $R$ be the set of all $2 \times 2$ matrices of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, where $a, b$ are integers.

(a) Show that $R$ is a subring of the ring of all $2 \times 2$ matrices with integer entries.

(b) Let $\mathbb{Z}[x]$ be the ring of polynomials in one indeterminate $x$, over the ring of integers $\mathbb{Z}$ and $I$ the principal ideal of $\mathbb{Z}[x]$ generated by $x^2 - 1$. Show that $R \cong \mathbb{Z}[x]/I$ (ring isomorphism).

Hint: Define a function $R \to \mathbb{Z}[x]/I$. 
III. FIELD THEORY

1.* Let $F$ be a subfield of a field $K$. Show that the following statements are equivalent.
   (a) Every irreducible polynomial $p(x) \in F[x]$ which has a root in $K$, splits in $K[x]$.
   (b) $K$ is the splitting field of a set of polynomials in $F[x]$.

2. Let $Q$ be the field of rationals and suppose $K$ is a splitting field of
   $(x^2 - 2)(x^2 + 1) \in Q[x]$.
   (a) What is $[K: Q]$?
   (b) Determine $G(K/Q)$.

3. (a) Define the characteristic of a field.
   (b) Define "prime field."
   (c) List all prime fields up to isomorphism and verify your result.

4. State the Fundamental Theorem of Galois Theory and define all
   relevant terms used.
1. Answer exactly nine questions. 
Select 5 from Part I and 4 from Part II.

2. When you are required to prove a particular statement, give as many details as time permits. In any case, be sure to give the main steps of the proof. In case of examples, an accurate description is sufficient.

3. The examination was checked carefully for misprints; if you think there is a mistake, do not interpret the question so as to make it trivial.

4. In some textbooks normal, regular and completely regular are defined to be Hausdorff. In this exam do not take any of these terms to imply Hausdorff. So, for example, if a problem requires you to prove a space is normal, you do not have to show it is Hausdorff.
Part I

1. Prove a product of completely regular spaces is completely regular.

2. Show every regular $T_0$-space is Hausdorff.

3. Prove or disprove: A sequence in the real line is convergent if it has exactly one cluster point.

4. Suppose $Y$ is a topological space, and $X_\alpha$ is a topological space for each $\alpha$ in an index set $A$. Suppose $f$ is a function from $Y$ into the product space $\prod\{X_\alpha : \alpha \in A\}$. Let $p_\alpha$ be the projection of the product space into $X_\alpha$. Show $\alpha f$ is continuous if and only if $p_\alpha \circ f$ is continuous for all $\alpha$.

5. Prove or disprove: If $X$ and $Y$ are topological spaces and $f$ is a closed, continuous function from $X$ onto $Y$ then $f$ is open.

6. (a) Show the following is false:

   If $X$ and $Y$ are topological spaces, $X = A \cup B$, $f$ is a function from $X$ to $Y$ and both $f|A$ and $f|B$ are continuous then $f$ is continuous.

   (b) What added hypothesis about $A$ and $B$ will make the statement in part (a) true? Prove your answer.

7. Prove or disprove:

   A closed subspace of a Lindelöf space is Lindelöf.

8. Suppose $X$ is a connected space, $p \in X$ and $X - \{p\} = A \cup B$ where $A$ and $B$ are separated. Show $A \cup \{p\}$ is connected.
PART II

1. Prove a compact subset of a Hausdorff space is closed.

2. Show a Tychonoff space $X$ is connected if and only if $\beta X$ is connected. ($\beta X$ is the Stone-Čech compactification of $X$.)

3. Prove: If $X$ is a Hausdorff space, $A \subseteq X$ and there is a continuous function $f: X \to A$ such that $f|A$ is the identity then $A$ is closed in $X$.

4. A space $X$ is countably compact if each countable open cover of $X$ has a finite subcover. Prove: If $X$ is countably compact and $\{F_n : n = 1, 2, 3, \cdots\}$ is a descending sequence of closed non-empty sets then $\cap \{F_n : n = 1, 2, 3, \cdots\}$ is non-empty.

5. Suppose $X$ is a topological space, $A \subseteq X$ and $p \in X$. $p$ is a limit point of $A$ if each neighborhood of $p$ contains a point of $A$ distinct from $p$. Prove: If $X$ is a second countable space and $A$ is an uncountable subset of $X$ then $A$ contains a limit point of itself.

6. Suppose $X$ and $Y$ are metric spaces and $f$ is a continuous function from $X$ into $Y$ such that $f^{-1}(K)$ is compact for each compact subset $K$ of $Y$. Show $f$ is closed.

7. Suppose $Y$ is a compact space and $f$ is a continuous function from $Y$ into itself. Show there is a closed set $A$ in $Y$ such that $f(A) = A$.

   (Hint: A form of the Axiom of Choice may be useful.)

8. Prove: A metric space $X$ is compact if and only if every real-valued function on $X$ is bounded.

9. Prove a countable product of metric spaces is metrizable.
The terms "holomorphic function" and "single-valued analytic function" are equivalent.

1. a. Find a conformal bijection of the extended complex plane which maps $0 \to 1$, $1 \to \infty$ and $\infty \to 0$. Remark that a conformal bijection is conformal, one-to-one and onto.

b. What is the image of the imaginary axis under that mapping?

2. a. Give a definition of the complex exponential function $e^z$, and show that $e^z$ is holomorphic in the complex plane.

b. Prove that $e^z$ is never zero in the complex plane and has an isolated essential singularity at $\infty$.

3. Let $w(z) = z^2 + z + 2$.

a. Where is $w(z)$ locally conformal? Remark that a function $f$ is locally conformal at a point $z$, if and only if there is a neighborhood $U$ of $z$ such that the restriction of $f$ to $U$ is defined, conformal and one-to-one.

b. Show that $w(z)$ is conformal in the disc $|z| < \frac{1}{2}$. Remark that a function $f$ is conformal in a domain $G$, if and only if $f$ is defined, conformal and one-to-one in $G$.

4. State and prove the so-called "fundamental theorem of algebra."

5. Let $\sum_{n=0}^{\infty} a_n z^n$ converge in the unit disc $D = \{z \text{ complex: } |z| < 1 \}$. Prove that $f(z) = \sum_{n=0}^{\infty} a_n z^n$, thus defined, is holomorphic in $D$, and that its derivative $f'(z)$ is equal to $\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$ in $D$.

6. Let $\varphi(\xi)$ be complex and continuous on a rectifiable curve $C$, represented by the complex function $\xi(t)$, defined and continuous on the real interval $0 \leq t \leq 1$.

a. Show that $f(z) = \int_C \frac{\varphi(\xi)}{\xi - z} \; d\xi$ is defined and holomorphic in each component of the complement of $C$.

b. Prove that $\lim_{z \to \infty} f(z) = 0$. 
7. Find and characterize all singularities of the complex trigonometric function \( \tan z \). In case you find poles or essential isolated singularities, calculate the corresponding residues.

8. a. State Rouche's Theorem.

b. Prove that if \( f(z) \) is holomorphic on \( E \) and \( |z| \leq 1 \) and if \( |f(z)| < 1 \) on \( E \), then the mapping \( w = f(z) \) has precisely one fixed point.
Real Variables Qualifying Examination: April 6, 1972

Instructions. a) Do at least one from each section.
b) Do no more than five problems.
c) While partial credit will be given for problems, complete solutions will be viewed, however, more favorably.

1.1. Suppose \( f \) is continuous on a bounded set \( E \) in the plane. Show that \( f \) can be extended to a continuous function \( F \) on the closure of \( E \) if and only if \( f \) is uniformly continuous on \( E \).

1.2. Let \( C^1[0, 1] \) denote the space of all functions \( f \) on \([0, 1]\) to \( \mathbb{R} \) that have continuous derivatives on \([0, 1]\). For \( f \in C^1[0, 1] \), let \( \|f\| = \sup_{0 \leq x \leq 1} |f(x)| + \sup_{0 \leq x \leq 1} |f'(x)| \). Show that \( C^1[0, 1] \) is complete with this norm.

1.3. A metric space \((S, \rho)\) is well-chained iff \( \forall \ \epsilon > 0 \), and \( x, y \in S \), there exists \( x = x_0, x_1, \ldots, x_n = y \) in \( S \) with \( \rho(x_i, x_{i+1}) < \epsilon \), \( 0 \leq i \leq n - 1 \). If \((S, \rho)\) is compact, prove that \( S \) is connected iff \( S \) is well-chained.

2.1. Suppose that \( E \) is a Lebesgue measurable subset of the reals, \( -\infty > m(E) > 0 \) and \( 0 < \delta < 1 \). Show that there exists a finite interval \( I \) such that \( m(E \cap I) > \delta \ m(E) \).

2.2. Let \( \{f_n\} \) be a sequence of real-valued functions, uniformly bounded and Riemann integrable on \([0, 1]\). Show that some subsequence of \( \{F_n\} \), \( F_n(x) = \int_0^x f_n(t) \, dt \) \( (0 \leq x \leq 1) \), converges uniformly on \([0, 1]\).

2.3. Suppose \( f \) is Lebesgue measurable on \([0, 1]\) and \( 0 \leq f(x) \leq 1 \) for almost all \( x \) in \([0, 1]\). Evaluate the limit \( \lim_{n \to \infty} \int_0^1 f^n(x) \, dx \).
3.1. Let \( f \in L^2[0, 1] \). Show that, for each \( n \geq 1 \), there is a unique \( p_n \), a polynomial of degree at most \( n \), of best approximation to \( f \) in \( \| \cdot \|_{L^2} \). Show also that \( \{ \| f - p_n \|_{L^2} \}_{n=1}^\infty \) is a non-increasing sequence converging to zero.

3.2. A point \( p \) in a convex set \( C \) is said to be an extreme point if \( p \) cannot be written as \( p = \alpha p_1 + (1 - \alpha)p_2 \) for \( p_1, p_2 \in C \), \( p_1 \neq p_2 \). Show that every \( x \) with norm 1 in a Hilbert space \( H \) is an extreme point of the unit ball in \( H \).

3.3. Let \( X_0 \) be a subspace of a Banach space \( X \) and \( T \) a bounded linear operator from \( X_0 \) into \( \ell^\infty \), the space of bounded sequences with the sup norm. Show that there exists a bounded linear extension \( \tilde{T} \) taking \( X \) into \( \ell^\infty \) such that \( \| \tilde{T} \| = \| T \| \). (HINT: Consider each coordinate.)