QUALIFYING EXAMINATION IN TOPOLOGY

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1. Answer exactly nine questions.
   Select 5 from Part I and
   5 from Part II.

2. When you are required to prove a particular statement, give as
   many details as time permits. In any case, be sure to give the
   main steps of the proof. In case of examples, an accurate
   description is sufficient.

3. The examination was checked carefully for misprints; if you think
   there is a mistake, do not interpret the question so as to make it
   trivial.

4. In some textbooks normal, regular and completely regular are
   defined to be Hausdorff. In this exam do not take any of these
   terms to imply Hausdorff. So, for example, if a problem requires
   you to prove a space is normal, you do not have to show it is
   Hausdorff.
PART I

1. Let \( \{ G_\alpha : \alpha \in A \} \) be a family of topologies for a set \( X \).
   
   (a) Show \( \bigcap \{ G_\alpha : \alpha \in A \} \) is a topology for \( X \).
   
   (b) Show \( \bigcup \{ G_\alpha : \alpha \in A \} \) may not be a topology for \( X \).

2. Let \( E \) be a subset of a metric space \((X, d)\). Show that the function
   
   \( f: X \to \mathbb{R} \) defined by \( f(x) = d(x, E) \) is continuous.

3. Let \( A \) and \( B \) be disjoint compact subsets of a Hausdorff space \( X \).
   
   Show that there exist disjoint open sets \( U \) and \( V \) containing \( A \)
   
   and \( B \), respectively.

4. Which of the following properties are hereditary?
   
   (a) separable
   
   (b) second countable
   
   (c) \( T_4 \) (normal plus \( T_2 \))

   Give a yes or no answer to all parts. Choose one part and justify
   
   your answer with a proof or counterexample.

5. A space \( X \) is called \( \sigma \)-compact iff \( X \) can be written as a union of
   
   countably many compact subsets. Show that every \( \sigma \)-compact space
   
   is Lindelöf.

6. Let \( X \) be a connected topological space, \( Y \) a connected subset and
   
   \( X - Y = A \cup B \), where \( A \) and \( B \) are separated. Show that \( A \cup Y \) is
   
   connected.

7. A subset \( E \) of a space \( X \) is called a retract of \( X \) iff there
   
   exists a continuous mapping \( r: X \to E \) such that \( r(x) = x \) for each
   
   \( x \in E \). Show that a retract of a Hausdorff space \( X \) is closed in \( X \).

8. Prove or give a counterexample: A product of regular spaces is
   
   regular.
PART II

1. A subset $E$ of a space $X$ is an $F_{\sigma}$ iff $E$ is a countable union of closed subsets of $X$. Prove: If $X$ is a regular space with a $\sigma$-locally finite base then every open subset of $X$ is an $F_{\sigma}$.

2. Prove: If every open subspace of a space $X$ is paracompact then $X$ is hereditarily paracompact.

3. Let $D$ be a decomposition of a space $X$ having only a finite number of non-degenerate elements, each of which is closed in $X$. Prove that $D$ is upper semicontinuous.

4. Let $X$ and $Y$ be topological spaces, $A$ and $B$ compact subsets of $X$ and $Y$, respectively, and $W$ a neighborhood of $A \times B$ in the product space $X \times Y$. Show that there are neighborhoods $U$ of $A$ and $V$ of $B$ such that $U \times V \subseteq W$.

5. Let $G$ be a family of closed compact sets such that $\bigcap \{A: A \in G\}$ is a subset of an open set $U$. Show that there is a finite subfamily $F$ of $G$ such that $\bigcap \{A: A \in F\} \subseteq U$.

6. A space $X$ is called completely normal iff given two separated subsets $A$ and $B$ of $X$, there exist disjoint open sets $U$ and $V$ containing $A$ and $B$, respectively. Prove: A space $X$ is completely normal iff every subspace of $X$ is normal.

7. (a) State the Nagata-Smirnov metrization theorem.

   (b) Prove: A compact Hausdorff space $X$ is metrizable iff $X$ is second-countable.

8. Let $X$ be a metrizable space. Show that the following are equivalent:

   (a) $X$ is second-countable

   (b) $X$ is separable

   (c) $X$ is Lindelöf.