# QUALIFYING EXAMINATION IN PROBABILITY

## Directions

**Your Name (printed, NO signature):** ______________________________

- Answer questions completely: a fully-done problem is far more revealing of your understanding than two half-done attempts.
- State your reasons for your claims. You may cite and use standard results. However, you have to demonstrate that you know why things are so; you cannot assume that we know that you know. That's the purpose of this examination! An exercise that states a standard result of the theory requires that you supply a proof.
- My best wishes to all of you.

1. (15 points) ______________________________
2. (15 points) ______________________________
3. (20 points) ______________________________
4. (15 points) ______________________________
5. (15 points) ______________________________
6. (15 points) ______________________________
7. (15 points) ______________________________

**TOTAL (110 points)** [      ]

**Problem 1**

Let $X$ be a nonnegative random variable, $X \geq 0$, and let $p > 0$. Show that

$$\mathbb{E}X^p = \int_0^\infty px^{p-1}\mathbb{P}[X > x]dx$$

. (Hint: start with the rhs and write $\mathbb{P}[X > x]$ as an integral).

**Problem 2**

(a) State both parts of the Borel-Cantelli Lemma carefully and prove one part.

(b) Give a simple example showing that the converse of Part I of Borel-Cantelli Lemma is trivially false.

(c) As a typical application of Part II of Borel-Cantelli Lemma, prove the following: If $X_1, X_2, \ldots$ are i.i.d. with $\mathbb{E}|X_i| = \infty$, then $\mathbb{P}[|X_n| \geq n \text{ i.o.}] = 1$.

**Problem 3**

Let $\{B_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, starting at 0.

(a) Define

$$\Xi_t = \exp\left(B_t - \frac{1}{2}t\right); \qquad t \ge 0.$$

Use Itô's formula to find an expression for $\Xi_t$ for $t > 0$.

(b) Rewrite your answer in part (a) as a simple SDE (together with the appropriate initial condition).

(c) Is there a probability measure $\tilde{\mathbb{P}}_T$ such that $\{B_t - t, \mathcal{F}_t; 0 \le t \le T\}$ is a BM on $(\Omega, \mathcal{F}, \tilde{\mathbb{P}}_T)$? If so, how is $\tilde{\mathbb{P}}_T$ defined?

(d) What can you say about $\mathbb{E}_{\mathbb{P}}\Xi_T$ and $\{\Xi_t, \mathcal{F}_t; 0 \le t \le T\}$? Why?

**Problem 4**

(a) State Lévy's continuity theorem for characteristic functions (c.f.'s).

(b) Assume that $X, Y$ are independent Poisson random variables; assume further that the distribution of $X$ has parameters $\lambda$ and the distribution of $Y$ has parameters $\mu$ ($X \stackrel{d}{=} Poi(\lambda)$ and $Y \stackrel{d}{=} Poi(\mu)$). Find the distribution of $Z = X + Y$.

**Problem 5**

(a) Define convergence in distribution, in probability, in $L^p$, and almost surely; and discuss relations between these types of convergence.

(b) Give a simple example of a sequence of random variables $\{X_n\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \to 0$ a.s. as $n \to \infty$ but for each $p \in (0, \infty]$, the sequence $\{X_n\}$ fails to converge in $L^p$.

**Problem 6**

(a) State the simplest form of the WLLN, SLLN, and CLT.

(b) Suppose $X_1, X_2, \ldots$ are iid random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}[X_1 = 1] = p = 1 - \mathbb{P}[X_1 = -1]$ and let $S_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$, and $S_0 = 0$. Show that if $p \neq \frac{1}{2}$ then state 0 must be transient for the random walk $\{S_n\}$.

**Problem 7**

Let $X$ be a discrete time Markov chain on $\{1, 2, 3, 4\}$ with one step transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Classify the states $\{1, 2, 3, 4\}$

(b) Find $\mathbf{P}^\infty := \lim_{n\to\infty} \mathbf{P}^n$ if you can, and state your observations about the long time behavior of this chain.