QUALIFYING EXAMINATION IN PROBABILITY

Directions

Your Name (printed, NO signature): ____________________________

- Answer questions completely: a fully-done problem is far more revealing of your understanding than two half-done attempts.

- State your reasons for your claims. You may cite and use standard results. However, you have to demonstrate that you know why things are so; you cannot assume that we know that you know. That’s the purpose of this examination! An exercise that states a standard result of the theory requires that you supply a proof.

- My best wishes to all of you.

1. (15 points) ____________________________________________________________________

2. (15 points) ____________________________________________________________________

3. (20 points) ____________________________________________________________________

4. (15 points) ____________________________________________________________________

5. (15 points) ____________________________________________________________________

6. (15 points) ____________________________________________________________________

7. (15 points) ____________________________________________________________________

TOTAL (110 points) ____________________________
Problem 1

Let $X$ be a nonnegative random variable, $X \geq 0$, and let $p > 0$. Show that

$$E X^p = \int_0^\infty px^{p-1}P[X > x]dx$$

. (Hint: start with the rhs and write $P[X > x]$ as an integral).
Problem 2

(a) State both parts of the Borel-Cantelli Lemma carefully and prove one part.

(b) Give a simple example showing that the converse of Part I of Borel-Cantelli Lemma is trivially false.

(c) As a typical application of Part II of Borel-Cantelli Lemma, prove the following:
   If $X_1, X_2, \ldots$ are i.i.d. with $\mathbb{E}|X_i| = \infty$, then $\mathbb{P}[|X_n| \geq n \text{ i.o.}] = 1$. 
Problem 3

Let \( \{B_t, \mathcal{F}_t; 0 \leq t < \infty\} \) be a Brownian motion on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), starting at 0.

(a) Define
\[
\Xi_t = \exp \left( B_t - \frac{1}{2} t \right); \quad t \geq 0.
\]
Use Itô’s formula to find an expression for \( \Xi_t \) for \( t > 0 \).

(b) Rewrite your answer in part (a) as a simple SDE (together with the appropriate initial condition).

(c) Is there a probability measure \( \tilde{\mathbb{P}}_T \) such that \( \{B_t - t, \mathcal{F}_t; 0 \leq t \leq T\} \) is a BM on \((\Omega, \mathcal{F}, \tilde{\mathbb{P}}_T)\)? If so, how is \( \tilde{\mathbb{P}}_T \) defined?

(d) What can you say about \( \mathbb{E}_T \Xi_T \) and \( \{\Xi_t, \mathcal{F}_t; 0 \leq t \leq T\} \)? Why?
Problem 4

(a) State Lévy’s continuity theorem for characteristic functions (c.f.’s).

(b) Assume that \( X, Y \) are independent Poisson random variables; assume further that the distribution of \( X \) has parameters \( \lambda \) and the distribution of \( Y \) has parameters \( \mu \) \((X \overset{d}{=} Poi(\lambda) \text{ and } Y \overset{d}{=} Poi(\mu))\). Find the distribution of \( Z = X + Y \).
Problem 5

(a) Define convergence in distribution, in probability, in \( L^p \), and almost surely; and discuss relations between these types of convergence.

(b) Give a simple example of a sequence of random variables \( \{X_n\} \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) such that \( X_n \to 0 \) a.s. as \( n \to \infty \) but for each \( p \in (0, \infty] \), the sequence \( \{X_n\} \) fails to converge in \( L^p \).
Problem 6

(a) State the simplest form of the WLLN, SLLN, and CLT.

(b) Suppose $X_1, X_2, \ldots$ are iid random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}[X_1 = 1] = p = 1 - \mathbb{P}[X_1 = -1]$ and let $S_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$, and $S_0 = 0$. Show that if $p \neq \frac{1}{2}$ then state 0 must be transient for the random walk $\{S_n\}$.
Problem 7

Let $X$ be a discrete time Markov chain on $\{1, 2, 3, 4\}$ with one step transition matrix

$$
P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 1
\end{bmatrix}$$

(a) Classify the states $\{1, 2, 3, 4\}$

(b) Find $P^\infty := \lim_{n \to \infty} P^n$ if you can, and state your observations about the long time behavior of this chain.